

Spectral Analysis of a Discrete Metastable System Driven by Lévy Flights

Toralf Burghoff* and Ilya Pavlyukevich†

Abstract

In this paper we consider a finite state time discrete Markov chain that mimics the behaviour of solutions of the stochastic differential equation

$$X_t^\varepsilon(x) = x - \int_0^t U'(X_s^\varepsilon) ds + \varepsilon L_t,$$

where U is a multi-well potential with $n \geq 2$ local minima and $L = (L_t)_{t \geq 0}$ is a symmetric α -stable Lévy process (Lévy flights process). We investigate the spectrum of the generator of this Markov chain in the limit $\varepsilon \rightarrow 0$ and localize the top n eigenvalues $\lambda_1^\varepsilon, \dots, \lambda_n^\varepsilon$. These eigenvalues turn out to be of the same algebraic order $\mathcal{O}(\varepsilon^\alpha)$ and are well separated from the rest of the spectrum by a spectral gap. We also determine the limits $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-\alpha} \lambda_i^\varepsilon$, $1 \leq i \leq n$, and show that the corresponding eigenvectors are approximately constant over the domains which correspond to the potential wells of U .

Keywords: metastability; Lévy flights; α -stable Lévy process; eigenvalues; spectral gap; Markov chain; transition times; fractional Laplacian; semiclassical limit.

AMS Subject Classification: 60J10*, 60J75, 47A75, 47G20, 15B51, 15A18

1 Introduction. Spectral properties of Gaussian small noise diffusions

Let $U \in C^3(\mathbb{R}, \mathbb{R})$ be a one-dimensional multi-well potential (see, e.g. Fig. 1) such that $|U'(x)| > c|x|^{1+c}$ as $x \rightarrow \pm\infty$ for some $c > 0$. Assume that U has exactly n local minima \mathfrak{m}_i , $1 \leq i \leq n$, as well as $n-1$ local maxima \mathfrak{s}_i , $1 \leq i \leq n-1$, enumerated in increasing order

$$-\infty = \mathfrak{s}_0 < \mathfrak{m}_1 < \mathfrak{s}_1 < \mathfrak{m}_2 < \dots < \mathfrak{m}_n < \mathfrak{s}_n = \infty. \quad (1.1)$$

Moreover, we suppose that all these extrema are non-degenerate, i.e.

$$U''(\mathfrak{m}_i) > 0, \quad 1 \leq i \leq n, \quad \text{and} \quad U''(\mathfrak{s}_i) < 0, \quad 1 \leq i \leq n-1, \quad (1.2)$$

and denote

$$\mathcal{M} := \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}. \quad (1.3)$$

The points \mathfrak{m}_i , $i = 1, \dots, n$, are exponentially stable attractors of a deterministic dynamical system generated by the ordinary differential equation $\dot{x}_t = -U'(x_t)$. For any initial position $x \in (\mathfrak{s}_{i-1}, \mathfrak{s}_i)$, the solution $x_t(x)$ does not leave the domain of attraction $(\mathfrak{s}_{i-1}, \mathfrak{s}_i)$ and $x_t(x) \rightarrow \mathfrak{m}_i$ as $t \rightarrow \infty$.

Although deterministic gradient systems with multiple attractors appear naturally in various application problems (e.g. climate modelling [2, 17, 38], physics [32], robotics [12], economics [45]), their evident

*Institute of Medical Biometry and Statistics, University of Lübeck, Ratzeburger Allee 160, Geb. 24, 23562 Lübeck Germany; toralf.burghoff@imbs.uni-luebeck.de

†Institute for Mathematics, Friedrich Schiller University Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany; ilya.pavlyukevich@uni-jena.de

drawback often consists in an impossibility of transitions between the domains of attraction. To overcome this limitation, a small noise is often added to the model. Intuitively the perturbed random system stays in the vicinity of one of the attractors \mathfrak{m}_i most of the time, making sporadic transitions to the other domains of attraction. This type of behaviour is often referred to as *metastability*. The essential characteristics of the small noise system as a metastable hopping process are the life times in the domains of attraction, transition probabilities between the wells, the most probable state etc. Naturally, these characteristics depend heavily on the type of the noisy perturbation. The case of Brownian perturbations is the most well studied.

For a standard Brownian motion $W = (W_t)_{t \geq 0}$, consider the SDE

$$X_t^\varepsilon(x) = x - \int_0^t U'(X_s^\varepsilon) ds + \varepsilon W_t. \quad (1.4)$$

Under the conditions on U formulated above, for any $x \in \mathbb{R}$, (1.4) has a unique strong solution $X^\varepsilon = (X_t^\varepsilon)_{t \geq 0}$ (see e.g. [14]). Since the classical results by Kramers [32], it is known that the mean life times of X^ε in the potential wells are exponentially long with respect to the parameter ε , more precisely of the order $\min\{e^{2(U(\mathfrak{s}_{i-1})-U(\mathfrak{m}_i))/\varepsilon^2}, e^{2(U(\mathfrak{s}_{i+1})-U(\mathfrak{m}_i))/\varepsilon^2}\}$ for the i -th well. Considering a generic perturbed gradient system on exponentially long time scales of the order $T(\varepsilon) = e^{\mu/\varepsilon^2}$, $\mu > 0$, one can recover different behaviours. In particular on the slowly growing (short) scales, when e^{μ/ε^2} is less than the life times in all the wells ($\mu < 2(U(\mathfrak{s}_{i+1})-U(\mathfrak{m}_i))$, $i = 1, \dots, n$), the trajectory $(X_{T(\varepsilon) \cdot t}^\varepsilon)_{t \in [0,1]}$ does not succeed in leaving any of the domains of attraction. On longer time scales, the trajectory may leave some of the domains of attraction thus making transitions and eventually forming a sublimit distribution. On the longest scale, X^ε converges to its stationary law which is concentrated in the vicinity of the potential's global minimum. For a generic potential, there are $n-1$ such critical values which are different and can be ordered in decreasing order $\mu_2 > \mu_3 > \dots > \mu_n$. These values determine the time scales $T(\varepsilon) = e^{\mu_i/\varepsilon^2}$ which distinguish the sublimit distributions. The values μ_i , $i = 2, \dots, n$, can be calculated probabilistically in terms of the heights of the potential barriers of U with the help of the large deviations theory by Freidlin and Wentzell (see [21, §VI.6] or a recent exposition [13]).

Recall that X^ε is a strong Markov processes with the infinitesimal generator

$$\mathcal{L}_\varepsilon f(x) = \frac{\varepsilon^2}{2} f''(x) - U'(x) f'(x), \quad f \in C_0^\infty(\mathbb{R}, \mathbb{R}). \quad (1.5)$$

The existence of multiple sublimit distributions is reflected in the spectral properties of the operator \mathcal{L}_ε . The generator \mathcal{L}_ε is a negative definite essentially self-adjoint operator in the weighted space $L^2(\mathbb{R}, e^{-2U(x)/\varepsilon^2} dx)$. It has a discrete non-positive spectrum $0 \geq -\lambda_1^\varepsilon \geq -\lambda_2^\varepsilon \geq \dots$ and a corresponding sequence of eigenfunctions $\{\Phi_i^\varepsilon\}_{i \geq 1}$ which form an orthonormal basis (see [24, Chapter 3] for the detailed exposition). Whereas its top eigenvalue is identically zero, $\lambda_1^\varepsilon \equiv 0$, with the corresponding eigenfunction being a constant, it turns out that the next $n-1$ eigenvalues are of the order $e^{-\mu_i/\varepsilon^2}$ where the rates μ_i are obtained above. The principle non-zero eigenvalue λ_n^ε corresponding to the longest time scale e^{μ_2/ε^2} determines the speed of convergence of the law of X_t^ε to the stationary distribution as $t \rightarrow \infty$; further eigenvalues recover the speeds of convergence to sublimit distributions. With the help of the Fourier method of separation of variables, the functional $u_\varepsilon(t, x) = \mathbb{E}_x f(X_t^\varepsilon)$ which is the solution of the Cauchy problem $\partial_t u = \mathcal{L}_\varepsilon u$, $u_\varepsilon(0, x) = f(x)$, can be represented as

$$u_\varepsilon(t, x) = \frac{\int_{\mathbb{R}} f(y) e^{-2U(y)/\varepsilon^2} dy}{\int_{\mathbb{R}} e^{-2U(y)/\varepsilon^2} dy} + \sum_{i=2}^n \left(\int_{\mathbb{R}} f(y) \Phi_i^\varepsilon(y) e^{-2U(y)/\varepsilon^2} dy \right) \cdot \Phi_i^\varepsilon(x) e^{-\lambda_i^\varepsilon t} + R_\varepsilon(t, x). \quad (1.6)$$

The remainder term R_ε can be shown to be negligible compared to the leading n terms of the expansion due to the spectral gap property, namely, that there is a constant M not depending on ε such that $\lambda_{n+1}^\varepsilon \geq M$ for all $\varepsilon \in (0, 1]$. Moreover, the eigenfunctions Φ_i^ε , $2 \leq i \leq n$, are almost constant over the potential wells.

A rich literature is devoted to the analysis of metastability of the system of the type (1.4) on different levels of rigour. A probabilistic characterization of the principle eigenvalue of \mathcal{L}_ε in a bounded domain with the Dirichlet boundary conditions was obtained by Khasminski [28] (see also Friedman [22, Lemma 1.1]). Schuss *et al.* [35, 42, 43] determined the shape of the eigenfunction Φ_2^ε and the asymptotics of the eigenvalue λ_2^ε , especially its subexponential prefactor in terms of the values $U''(\mathfrak{s}_i)$ and $U''(\mathfrak{m}_i)$ with the help of formal asymptotic expansions of solutions of second order ordinary differential equations. Buslov,

Makarov and Kolokoltsov [10, 11, 30, 31, 34] established a connection between the top spectrum of the diffusion's generator and the spectrum of the matrix of the inverse mean life times of X^ε in the potential wells, found very exact approximations for the corresponding eigenfunctions and proved the existence of an ε -independent spectral gap. Bovier, Eckhoff, Gaynard and Klein in [6–8, 20] developed a potential theoretic approach to metastability of Markov chains and gradient diffusions and obtained very exact asymptotics of the top eigenvalues. Metastable behaviour of diffusions in a double-well potential was studied by probabilistic methods in [23, 29, 39]. Applications of the spectral theory to simulated annealing can be found in [16, 25, 26], to stochastic resonance in [24], and to analysis of molecular dynamics [37, 44]. Berglund and Gentz [4] studied the behaviour of small noise diffusions in potentials with non-quadratic extrema. We refer the reader to a recent review [3] by Berglund for further references on the subject.

In recent time, equations driven by non-Gaussian noise, especially α -stable Lévy processes (Lévy flights), are being adopted for description and modelling of various real world phenomena (see e.g. [36]). Let $L = (L_t)_{t \geq 0}$ be a symmetric α -stable Lévy process with the characteristic function

$$\mathbb{E} e^{iuL_t} = e^{-c(\alpha)t|u|^\alpha}, \quad u \in \mathbb{R}, \quad \alpha \in (0, 2), \quad (1.7)$$

where $c(\alpha) = \alpha \int_0^\infty \frac{1 - \cos y}{y^{1+\alpha}} dy \in (0, \infty)$. Such a process has heavy tails and for convenience the constant $c(\alpha)$ is chosen to guarantee the following tail asymptotics:

$$\lim_{u \rightarrow \infty} u^\alpha \mathbb{P}(|L_1| \geq u) = 1. \quad (1.8)$$

For $\varepsilon > 0$, the SDE

$$X_t^\varepsilon(x) = x - \int_0^t U'(X_s^\varepsilon) ds + \varepsilon L_t \quad (1.9)$$

possesses a unique strong solution $X^\varepsilon = (X_t^\varepsilon)_{t \geq 0}$ which is a strong Markov processes (see [1, 27]).

In the limit $\varepsilon \rightarrow 0$, the process X^ε enjoys the metastable behaviour in the following sense.

Theorem 1.1 (Theorem 1.1, [27]). *Let $k = 1, \dots, n$ and $x \in (\mathfrak{s}_{k-1}, \mathfrak{s}_k)$. Then the process $(X_{t/\varepsilon^\alpha}^\varepsilon)_{t > 0}$, $X_0^\varepsilon = x$, converges, as $\varepsilon \rightarrow 0$, in the sense of finite dimensional distributions to a Markov chain $Y = (Y_t)_{t > 0}$, $Y_0 = \mathfrak{m}_k$, on the state space \mathcal{M} with the stable conservative generator $\mathbf{Q} = (q_{i,j})_{i,j=1}^n$ given by*

$$q_{i,j} = \begin{cases} \frac{1}{2} \left| \frac{1}{|\mathfrak{s}_{j-1} - \mathfrak{m}_i|^\alpha} - \frac{1}{|\mathfrak{s}_j - \mathfrak{m}_i|^\alpha} \right|, & i \neq j, \\ -\frac{1}{2} \left(\frac{1}{|\mathfrak{s}_{i-1} - \mathfrak{m}_i|^\alpha} + \frac{1}{|\mathfrak{s}_i - \mathfrak{m}_i|^\alpha} \right), & i = j. \end{cases} \quad (1.10)$$

■

In other words, there is a time scale $T(\varepsilon) = \varepsilon^{-\alpha}$ on which the process X^ε reminds of a finite state Markov chain Y . It was also shown in [27], that on slower time scales $\varepsilon^{-\mu}$, $\mu < \alpha$, the process X^ε does not leave the potential well where it has started. On faster time scales $\varepsilon^{-\mu}$, $\mu > \alpha$, one cannot obtain a meaningful limit of X^ε since its life times in the potential wells converge to zero and the process $X_{t/\varepsilon^\alpha}^\varepsilon$ persistently jumps between different wells.

The generator of X^ε is the integro-differential operator

$$\begin{aligned} \mathcal{D}_\varepsilon f(x) &= \frac{\alpha}{2} \varepsilon^\alpha \int_0^\infty \frac{f(x+z) - 2f(x) + f(x-z)}{z^{1+\alpha}} dz - U'(x)f'(x) \\ &= -\frac{\alpha}{2} \varepsilon^\alpha (-\Delta)^{\alpha/2} f(x) - U'(x)f'(x), \quad f \in C_0^\infty(\mathbb{R}, \mathbb{R}), \end{aligned} \quad (1.11)$$

and the metastability result of Theorem 1.1 suggests that the top spectrum of \mathcal{D}_ε should remind of the spectrum of the Markov chain Y . One can expect that the top n eigenvalues $\lambda_1^\varepsilon = 0, \lambda_2^\varepsilon, \dots, \lambda_n^\varepsilon$ of \mathcal{D}_ε should be closely connected with the eigenvalues of the matrix \mathbf{Q} and well separated from the rest of the spectrum by a spectral gap. Since the time scale in the metastability result is unique, the eigenvalues should be all of the same order $\mathcal{O}(\varepsilon^\alpha)$.

Unfortunately not much can be said about the spectrum of \mathcal{D}_ε . No weighted space is known where \mathcal{D}_ε is self-adjoint and no general spectral theory can be applied in this case. Although it is known that the invariant distribution exists and is unique [33, 41], explicit formulae for its density can be obtained only in a very few particular cases (mainly for the Cauchy process $\alpha = 1$ and polynomial potentials, see [15, 18, 19]).

In this paper we make the first step towards a better understanding of the spectral properties of X^ε by reduction of a jump-diffusion to a finite state discrete time Markov chain and analysing its spectral properties in the limit $\varepsilon \rightarrow 0$.

2 A discrete Lévy driven system and the main result

We construct a discrete-time Markov chain on a finite state space that mimic the metastable behaviour of the solution X^ε of (1.9) in the limit of small ε . The construction is based on the standard Euler scheme for SDEs.

For any $\gamma > 0$ there are a range parameter $R > 0$, a time step $h > 0$ as well as the spacial mesh parameter $\delta > 0$ such that $R^{-1} + h + \delta \leq \gamma$ and such that the following holds true.

All the local minima of U belong to $[-R, R]$, $-R < \mathfrak{m}_1 < \mathfrak{m}_n < R$. With R fixed, let us redefine the points $\mathfrak{s}_0 := -R$ and $\mathfrak{s}_n := R$ for convenience.

There exists a finite set of points \mathcal{S} and a partition $\bigsqcup_{x \in \mathcal{S}} I_x = [\mathfrak{s}_0, \mathfrak{s}_n]$ consisting of the intervals $I_x = [a_x, b_x]$ such that $\max_{x \in \mathcal{S}} |b_x - a_x| \leq \delta/2$, each interval I_x contains only one point from \mathcal{S} and $x \in (a_x, b_x)$, $\mathcal{M} \subset \mathcal{S}$, $\{\mathfrak{s}_0, \dots, \mathfrak{s}_n\} \subset \{a_x, b_x\}_{x \in \mathcal{S}}$, and for any $x \in \mathcal{S} \setminus \mathcal{M}$

$$x - hU'(x) \notin I_x. \quad (2.1)$$

Example 2.1. The set \mathcal{S} and the intervals I_x can be constructed as follows. Let $R > \frac{3}{\gamma} \vee |\mathfrak{m}_1| \vee |\mathfrak{m}_n|$. Let $\delta < \frac{\gamma}{3} \wedge \frac{1}{8} \min_{i,j} |\mathfrak{s}_i - \mathfrak{m}_j|$.

First, we demand that all $\mathfrak{m}_i \in \mathcal{S}$ and set $I_{\mathfrak{m}_i} = [\mathfrak{m}_i - \frac{\delta}{4}, \mathfrak{m}_i + \frac{\delta}{4}]$. Choose $0 < h \leq \gamma/3$ such that for all $i = 1, \dots, n$, all $x \in [\mathfrak{s}_{i-1} + \frac{\delta}{4}, \mathfrak{m}_i - \frac{\delta}{4}]$

$$x - U'(x)h \in [\mathfrak{s}_{i-1}, \mathfrak{m}_i] \quad (2.2)$$

and for $x \in [\mathfrak{m}_i + \frac{\delta}{4}, \mathfrak{s}_i - \frac{\delta}{4}]$

$$x - U'(x)h \in [\mathfrak{m}_i, \mathfrak{s}_i]. \quad (2.3)$$

For definiteness, let us construct the partition of the interval $[\mathfrak{m}_1 + \frac{\delta}{4}, \mathfrak{s}_1]$. Let z_1 be the solution of the equation $z - U'(z)h = \mathfrak{m}_1 + \frac{\delta}{4}$, $z_1 > \mathfrak{m}_1 + \frac{\delta}{4}$. Decompose the interval

$$[\mathfrak{m}_1 + \frac{\delta}{4}, z_1 \wedge \mathfrak{s}_1 - \frac{\delta}{4}] \quad (2.4)$$

into a finite disjoint union of the intervals $[a_x, b_x]$, $b_x - a_x < \frac{\delta}{2}$, denote by x the middle point of each such an interval, and add x to \mathcal{S} . Then due to the fact that $U'(x) > 0$ on $[\mathfrak{m}_1 + \frac{\delta}{4}, \mathfrak{s}_1 - \frac{\delta}{4}]$, we get $x - U'(x)h \in I_{\mathfrak{m}_1}$. Denote z_2 the solution of the equation $z - U'(z)h = z_1$, $z_2 > z_1$ and again decompose the interval $[z_1, z_2 \wedge \mathfrak{s}_1 - \frac{\delta}{4}]$ a finite disjoint union of the intervals of the maximal length $\frac{\delta}{2}$. Continue this procedure and assign the last interval to be $(a_x, \mathfrak{s}_i]$. ■

For any $x \in \mathcal{S}$ let $y^*(x) \in \mathcal{S}$ be the unique element such that

$$x - hU'(x) \in I_{y^*(x)}. \quad (2.5)$$

Let us denote by $T: \mathcal{S} \rightarrow \mathcal{S}$ the mapping corresponding to the operation “*”, i.e. $Tx := y^*(x)$. The condition (2.1) implies that $Tx = y^*(x) \neq x$ for $x \in \mathcal{S} \setminus \mathcal{M}$ whereas $Tx = y^*(x) = x$ for $x \in \mathcal{M}$.

On the set \mathcal{S} define a discrete time deterministic motion $Z^0 = (Z_k^0)_{k \geq 0}$ such that

$$Z_0^0 = x, \quad Z_k^0 = y^*(Z_{k-1}^0) = T^k x, \quad k \geq 1. \quad (2.6)$$

For any $x \in \mathcal{S}$ the sequence $\{Z_k^0(x)\}_{k \geq 0}$ is monotone.

The deterministic motion Z^0 mimics the behaviour of the solutions of the deterministic ordinary differential equation $\dot{x}_t = -U'(x_t)$. It can also be described as a discrete time Markov chain on \mathcal{S} with the matrix $\mathbf{P}^0 = (p_{x,y}^0)_{x,y \in \mathcal{S}}$ of one step transition probabilities given by

$$p_{x,y}^0 := \begin{cases} 1, & y = y^*(x), \\ 0, & \text{otherwise.} \end{cases} \quad (2.7)$$

Eventually let us construct a discrete-time Markov chain $Z^\varepsilon = (Z_k^\varepsilon)_{k \geq 0}$ on the state space \mathcal{S} which mimics the behaviour of the jump-diffusion X^ε and can be considered as a random perturbation of Z^0 .

Denote $\tilde{y} := \min\{y: y \in \mathcal{S}\}$ and $\hat{y} := \max\{y: y \in \mathcal{S}\}$ and define the matrix of one-step transition probabilities $\mathbf{P}^\varepsilon = (p_{x,y}^\varepsilon)_{x,y \in \mathcal{S}}$ of Z^ε as

$$p_{x,y}^\varepsilon := \mathbb{P}(x - hU'(x) + \varepsilon h^{\frac{1}{\alpha}} L_1 \in I_y), \quad y \neq \tilde{y}, \hat{y}, \quad (2.8)$$

as well as

$$p_{x,\tilde{y}}^\varepsilon := \mathbb{P}(x - hU'(x) + \varepsilon h^{\frac{1}{\alpha}} L_1 \leq b_{\tilde{y}}) \quad (2.9)$$

$$p_{x,\bar{y}}^\varepsilon := \mathbb{P}(x - hU'(x_k) + \varepsilon h^{\frac{1}{\alpha}} L_1 \geq a_{\bar{y}}). \quad (2.10)$$

By construction, the Markov chain Z^ε gets reflected at the barriers $-R$ and R which mimics the very fast return from infinity of the trajectory of X^ε due to the fast increase of the potential at infinity.

First we have to study the metastable behaviour of Z^ε as $\varepsilon \rightarrow 0$. Since Z^ε is constructed in such a way that it resembles the Lévy driven jump diffusion X^ε one could expect that the Z^ε demonstrates the same metastable behaviour. Indeed, the following analogue of the Theorem 1.1 holds true.

Theorem 2.2. *Let $k = 1, \dots, n$ and $x \in \mathcal{S} \cap [\mathfrak{s}_{k-1}, \mathfrak{s}_k]$. Then the process $(Z_{\lfloor \frac{t}{h\varepsilon^\alpha} \rfloor}^\varepsilon)_{t>0}$, $Z_0^\varepsilon = x$, converges, as $\varepsilon \rightarrow 0$, in the sense of finite dimensional distributions to a Markov chain $Y = (Y_t)_{t>0}$, $Y_0 = \mathfrak{m}_k$, on the state space \mathcal{M} with the generator \mathbf{Q} defined in (1.10).*

Proof. The proof of this result essentially follows the arguments presented in [27] for the case of a one dimensional jump diffusion X^ε . The arguments are even easier since by construction, \mathcal{S} does not contain the local maxima of U . Furthermore thanks to the reflection condition at $\pm R$ and the finiteness of \mathcal{S} , we do not have to consider returns of Z^ε from infinity. Note that this convergence result does not depend on the size or the particular choice of the state space \mathcal{S} . For a detailed proof we refer the reader to Chapter 3 in [9]. \square

The Theorem 2.2 suggests that top spectrum of the generator of the Markov chain Z^ε should be closely related to the spectrum of the matrix \mathbf{Q} . Let $N = |\mathcal{S}|$ denote the cardinality of the set \mathcal{S} . To compare the generators of Z^ε and Y we note that

$$\mathbf{P}^\varepsilon = e^{\mathbf{P}^\varepsilon - \mathbf{I}_N}, \quad (2.11)$$

so that the matrix $\mathbf{P}^\varepsilon - \mathbf{I}_N$ can be viewed as the discrete analogue of an infinitesimal generator. Taking into account the time scaling $t \mapsto \frac{t}{h\varepsilon^\alpha}$ which appears in the metastability result (Theorem 2.2), we introduce the $N \times N$ matrix

$$\mathbf{Q}^\varepsilon := \frac{1}{h\varepsilon^\alpha} (\mathbf{P}^\varepsilon - \mathbf{I}_N) \quad (2.12)$$

and denote by $\sigma(\mathbf{Q}^\varepsilon) := \{\lambda_1^\varepsilon, \dots, \lambda_N^\varepsilon\}$ and $\sigma(\mathbf{Q}) := \{\lambda_1^\mathbf{Q}, \dots, \lambda_n^\mathbf{Q}\}$ the spectra of \mathbf{Q}^ε and \mathbf{Q} , respectively. Then the following main theorem holds.

Theorem 2.3. *The spectrum $\sigma(\mathbf{Q}^\varepsilon)$ can be divided into two disjoint parts $\sigma_1(\mathbf{Q}^\varepsilon)$ and $\sigma_2(\mathbf{Q}^\varepsilon)$ for which the following assertions hold:*

(i) $\sigma_1(\mathbf{Q}^\varepsilon)$ contains precisely n eigenvalues $\lambda_1^\varepsilon, \dots, \lambda_n^\varepsilon$ such that $\lambda_1^\varepsilon = \lambda_1^\mathbf{Q} = 0$ and

$$\lim_{\varepsilon \rightarrow 0} \lambda_i^\varepsilon = \lambda_i^\mathbf{Q}, \quad 2 \leq i \leq n; \quad (2.13)$$

(ii) For any sequence $\{r_\varepsilon\}_{\varepsilon>0}$ such that $r_\varepsilon \rightarrow +\infty$, $r_\varepsilon \varepsilon^{\frac{\alpha}{n+1}} \rightarrow 0$ there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $i = n+1, \dots, N$

$$|\lambda_i^\varepsilon| \geq r_\varepsilon. \quad (2.14)$$

The proof of the Theorem 2.3 does not take into account any a priori information about the eigenvalues of \mathbf{Q} other than the fact that one of them is 0. However, we strongly believe that in the generic case, e.g. when the potential U does not have any intrinsic symmetries, all eigenvalues of \mathbf{Q} are real and simple. This conjecture is supported by numerous computer simulations of spectra of a matrix \mathbf{Q} , when the entries of the form (1.10) are randomly generated.

Denote by $\delta^j = (\delta_x^j)_{x \in \mathcal{S}}$, $1 \leq j \leq n$, the indicator function (vector) of the set $\mathcal{S}_j := \mathcal{S} \cap [\mathfrak{s}_{j-1}, \mathfrak{s}_j]$, i.e. $\delta_x^j = 1$ for $x \in \mathcal{S}_j$ and 0 elsewhere.

Theorem 2.4. *Assume that the eigenvalues of \mathbf{Q} are real and simple. Let $1 \leq i \leq n$ and let $\psi^{\varepsilon,i} = (\psi_x^{\varepsilon,i})_{x \in \mathcal{S}}$ be the right eigenvector of \mathbf{Q}^ε associated with λ_i^ε and normalized such that $\max_{1 \leq j \leq n} |\psi_{\mathfrak{m}_j}^{\varepsilon,i}| = 1$. Let $\psi^{\mathbf{Q},i} = (\psi_j^{\mathbf{Q},i})_{j=1}^n$ be the right eigenvector of \mathbf{Q} associated with $\lambda_i^\mathbf{Q}$ and normalized such that $\max_{1 \leq j \leq n} |\psi_j^{\mathbf{Q},i}| = 1$. Then the following limit holds:*

$$\max_{x \in \mathcal{S}} \left| \psi_x^{\varepsilon,i} - \sum_{j=1}^n \psi_j^{\mathbf{Q},i} \delta_x^j \right| \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (2.15)$$

In particular, for $i = 1$, $\lambda_1^\varepsilon = 0$ and $\psi^{\varepsilon,1} = \mathbf{1}$.

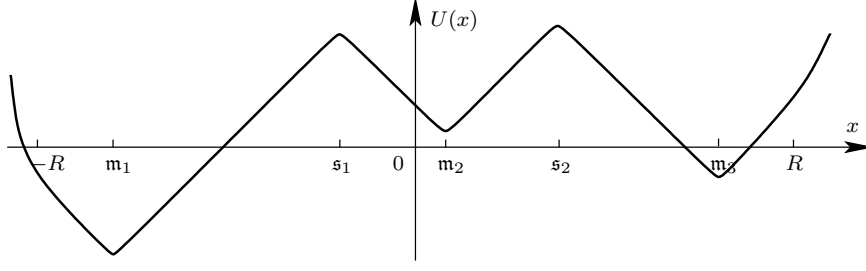


Figure 1: A potential U with the minima $m_1 = -4.015$, $m_2 = 0.468$, $m_3 = 3.966$ and the maxima $s_1 = -1.034$, $s_2 = 1.921$; the range $R = 5$.

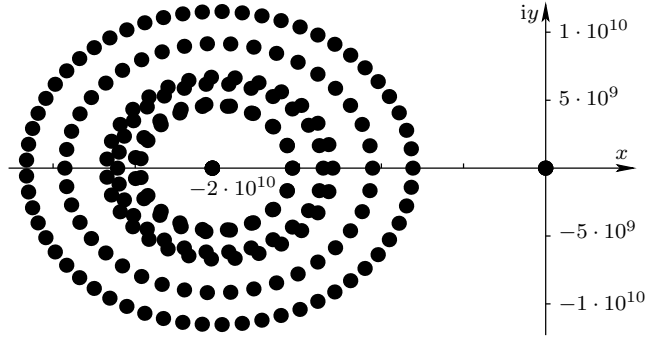


Figure 2: The spectrum of the matrix \mathbf{Q}^ϵ on the complex plane. The top three eigenvalues are located in the neighbourhood of the origin and well separated from the rest of the spectrum.

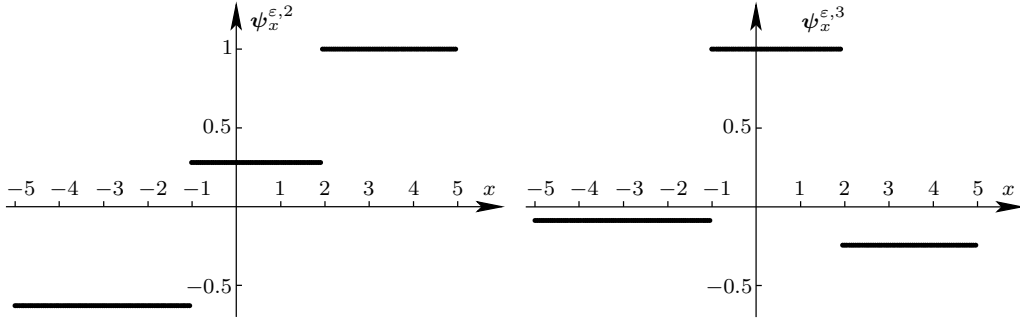


Figure 3: The eigenvectors $\psi^{\epsilon,2}$ and $\psi^{\epsilon,3}$ of the matrix \mathbf{Q}^ϵ are almost constant over the domains of attraction of U .

Example 2.5. Consider a three-well potential U such that, for simplicity of simulations, $U' = \pm 1$ outside of small neighbourhoods of the extrema, see Fig. 1, and fix $\alpha = 1.8$. The matrix \mathbf{Q} calculated according to (1.10) has the eigenvalues $\lambda_1^{\mathbf{Q}} = 0$, $\lambda_2^{\mathbf{Q}} = -0.124$, $\lambda_3^{\mathbf{Q}} = -0.579$, and the corresponding eigenvectors $\psi^{\mathbf{Q},1} = (1, 1, 1)$, $\psi^{\mathbf{Q},2} = (-0.629, 0.280, 1)$, $\psi^{\mathbf{Q},3} = (-0.088, 1, -0.245)$. We construct the state space \mathcal{S} consisting of $N = 203$ points from the interval $[-5, 5)$ and a Markov chain with the matrix of transition probabilities defined according to (2.8), (2.9), (2.10) with $h = 10/N$ and $\epsilon = 10^{-5}$. The spectrum of the matrix $\mathbf{Q}^\epsilon = \frac{1}{\epsilon^\alpha h}(\mathbf{P}^\epsilon - \mathbf{I}_N)$ consists of 203 eigenvalues, see Fig. 2, and the top eigenvalues are λ_1^ϵ , λ_2^ϵ and λ_3^ϵ , coincide with the spectrum of \mathbf{Q} with a four decimal places precision. The eigenvector $\psi^{\epsilon,1}$ is constant on \mathcal{S} whereas the eigenvectors $\psi^{\epsilon,2}$ and $\psi^{\epsilon,3}$ converge to the values $\psi_k^{\mathbf{Q},2}$, $\psi_k^{\mathbf{Q},3}$, $k = 1, 2, 3$, on the parts of \mathcal{S} corresponding to the domains of attraction of U , see Fig. 3.

3 Eigenvalues. Proof of the Theorem 2.3

Let L_1 by a symmetric α -stable random variable with the characteristic function (1.7). It is well-known (see [5]) that L_1 has a density $p_\alpha(x)$ on \mathbb{R} which can be expanded as $x \rightarrow +\infty$ as:

$$p_\alpha(x) = \frac{\alpha}{2x^{\alpha+1}} + \mathcal{O}\left(\frac{1}{x^{2\alpha+1}}\right). \quad (3.1)$$

In particular, for any fixed $0 < a < b$ and $h > 0$ there is $C > 0$ such that in the limit $\varepsilon \rightarrow 0$

$$\left| \mathbb{P}(a \leq \varepsilon h^{\frac{1}{\alpha}} L_1 < b) - \frac{\varepsilon^\alpha h}{2} \left(\frac{1}{a^\alpha} - \frac{1}{b^\alpha} \right) \right| \leq C \varepsilon^{2\alpha}. \quad (3.2)$$

The next Lemma follows directly from the formulae (2.8), (2.9) and (2.10).

Lemma 3.1. *Let $\alpha \in (0, 2)$ and the set \mathcal{S} and the partition $[-R, R] = \bigsqcup_{x \in \mathcal{S}} [a_x, b_x]$ be fixed. Then there is $\varepsilon_0 > 0$ small enough and a constant $C = C(h, \delta, R, \alpha) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following estimates hold:*

(i) *for any $x \in \mathcal{S}$ and any $y \in \mathcal{S}$ such that $y \neq y^*(x)$ we have*

$$|p_{x,y}^\varepsilon - d_{x,y} \varepsilon^\alpha h| \leq C \varepsilon^{2\alpha} \quad (3.3)$$

where

$$d_{x,y} = \begin{cases} \frac{1}{2} \left| \frac{1}{|a_y - x + hU'(x)|^\alpha} - \frac{1}{|b_y - x + hU'(x)|^\alpha} \right|, & y \notin \{\tilde{y}, \hat{y}, y^*(x)\}, \\ \frac{1}{2|b_{\tilde{y}} - x + hU'(x)|^\alpha}, & y = \tilde{y}, \\ \frac{1}{2|a_{\hat{y}} - x + hU'(x)|^\alpha}, & y = \hat{y}; \end{cases} \quad (3.4)$$

(ii) *for $x \in \mathcal{S}$ and $y = y^*(x)$ we have*

$$|p_{x,y^*(x)}^\varepsilon - (1 - f_{x,y^*(x)} \varepsilon^\alpha h)| \leq C \varepsilon^{2\alpha}, \quad (3.5)$$

where

$$f_{x,y^*(x)} = \frac{1}{2} \left(\frac{1}{|a_{y^*(x)} - x + hU'(x)|^\alpha} + \frac{1}{|b_{y^*(x)} - x + hU'(x)|^\alpha} \right). \quad (3.6)$$

3.1 Proof of Theorem 2.3

In the first step we will analyse the structure of the matrix

$$\mathbf{A}^\varepsilon(\lambda) := \mathbf{Q}^\varepsilon - \lambda \mathbf{I}_N = \frac{1}{\varepsilon^\alpha h} (\mathbf{P}^\varepsilon - \mathbf{I}_N) - \lambda \mathbf{I}_N, \quad \lambda \in \mathbb{C}, \quad (3.7)$$

which obviously plays the essential role in the investigation of the spectrum of \mathbf{Q}^ε .

First we observe that the matrices \mathbf{P}^ε and $\mathbf{A}^\varepsilon(\lambda)$ possess a block structure. Indeed, recalling the decomposition of the state space $\mathcal{S} = \mathcal{S}_1 \sqcup \dots \sqcup \mathcal{S}_n$, $\mathcal{S}_i = \mathcal{S} \cap [\mathfrak{s}_{i-1}, \mathfrak{s}_i]$, $i = 1, \dots, n$, as a disjoint union of states belonging to different potential wells, we may write

$$\mathbf{P}^\varepsilon = \begin{pmatrix} \mathbf{P}_{\mathcal{S}_1, \mathcal{S}_1}^\varepsilon & \cdots & \mathbf{P}_{\mathcal{S}_1, \mathcal{S}_n}^\varepsilon \\ \vdots & & \vdots \\ \mathbf{P}_{\mathcal{S}_n, \mathcal{S}_1}^\varepsilon & \cdots & \mathbf{P}_{\mathcal{S}_n, \mathcal{S}_n}^\varepsilon \end{pmatrix}, \quad \mathbf{A}^\varepsilon(\lambda) = \begin{pmatrix} \mathbf{A}_{\mathcal{S}_1, \mathcal{S}_1}^\varepsilon(\lambda) & \cdots & \mathbf{A}_{\mathcal{S}_1, \mathcal{S}_n}^\varepsilon \\ \vdots & & \vdots \\ \mathbf{A}_{\mathcal{S}_n, \mathcal{S}_1}^\varepsilon & \cdots & \mathbf{A}_{\mathcal{S}_n, \mathcal{S}_n}^\varepsilon(\lambda) \end{pmatrix}. \quad (3.8)$$

The blocks $\mathbf{P}_{\mathcal{S}_i, \mathcal{S}_j}^\varepsilon$ of \mathbf{P}^ε determine the transitions of the Markov chain Z^ε between the wells \mathcal{S}_i and \mathcal{S}_j . Only the blocks $\mathbf{A}_{\mathcal{S}_i, \mathcal{S}_i}^\varepsilon(\lambda)$, $i = 1, \dots, n$, that include the main diagonal depend on λ .

For the further analysis of the matrix $\mathbf{A}^\varepsilon(\lambda)$ it is helpful to study the limit behaviour of its entries $a_{x,y}^\varepsilon(\lambda)$ as $\varepsilon \rightarrow 0$.

The following expansions follow immediately from Lemma 3.1.

Lemma 3.2. *Let $\lambda \in \mathbb{C}$, then*

(i) *for $x, y \in \mathcal{S}$ and $y \notin \{x, y^*(x)\}$ we have $a_{x,y}^\varepsilon = d_{x,y}(1 + \mathcal{O}(\varepsilon^\alpha))$;*

(ii) for $x \in \mathcal{S} \setminus \mathcal{M}$ and $y = y^*(x)$ we have $a_{x,y^*(x)}^\varepsilon(\lambda) = \mathcal{O}(\varepsilon^{-\alpha})$, more precisely,

$$a_{x,y^*(x)}^\varepsilon(\lambda) = \frac{1}{\varepsilon^\alpha h} - \tilde{a}_{x,y^*(x)}^\varepsilon \quad \text{where} \quad \tilde{a}_{x,y^*(x)}^\varepsilon = f_{x,y^*(x)}(1 + \mathcal{O}(\varepsilon^\alpha)); \quad (3.9)$$

(iii) for $x \in \mathcal{S} \setminus \mathcal{M}$ we have $a_{x,x}^\varepsilon(\lambda) = \mathcal{O}(\varepsilon^{-\alpha})$, more precisely,

$$a_{x,x}^\varepsilon(\lambda) = -\frac{1}{\varepsilon^\alpha h} - \lambda + \tilde{a}_{x,x}^\varepsilon \quad \text{where} \quad \tilde{a}_{x,x}^\varepsilon = d_{x,x}(1 + \mathcal{O}(\varepsilon^\alpha)); \quad (3.10)$$

(iv) for $x \in \mathcal{M}$ we have $a_{x,x}^\varepsilon = \mathcal{O}(1)$, more precisely,

$$a_{x,x}^\varepsilon = -\tilde{a}_{x,x}^\varepsilon - \lambda \quad \text{where} \quad \tilde{a}_{x,x}^\varepsilon = f_{x,x}(1 + \mathcal{O}(\varepsilon^\alpha)). \quad (3.11)$$

As in Lemma 3.1, the bounds hidden in the Landau symbols \mathcal{O} are uniform over $x, y \in \mathcal{S}$.

To get a better understanding of the structure of $\mathbf{A}^\varepsilon(\lambda)$, let us for example take a detailed look at the block $\mathbf{A}_{\mathcal{S}_1, \mathcal{S}_1}^\varepsilon(\lambda)$. If we number the states of \mathcal{S}_1 in the increasing order by $x_1, \dots, x_{N(1)}$, then this block has the form

$$\begin{pmatrix} -\frac{1}{\varepsilon^\alpha h} - \lambda + \tilde{a}_{x_1, x_1}^\varepsilon & \cdots & \frac{1}{\varepsilon^\alpha h} - \tilde{a}_{x_1, y^*(x_1)}^\varepsilon & \cdots & a_{x_1, m_1}^\varepsilon & \cdots & a_{x_1, x_{N(1)}}^\varepsilon \\ \vdots & & \vdots & & \cdots & & \vdots \\ a_{m_1, x_1}^\varepsilon & & a_{m_1, y^*(x_1)}^\varepsilon & & -\tilde{a}_{m_1, m_1}^\varepsilon - \lambda & & a_{m_1, x_{N(1)}}^\varepsilon \\ \vdots & & \vdots & & \cdots & & \vdots \\ a_{x_{N(1)}, x_1}^\varepsilon & & a_{x_{N(1)}, y^*(x_1)}^\varepsilon & & a_{x_{N(1)}, m_1}^\varepsilon & \cdots & -\frac{1}{\varepsilon^\alpha h} - \lambda + \tilde{a}_{x_{N(1)}, x_{N(1)}}^\varepsilon \end{pmatrix} \quad (3.12)$$

Note that the numbers $\tilde{a}_{x,y}^\varepsilon$ that appear in the previous lemma are all positive.

Let us establish the connection between the matrix $\frac{1}{\varepsilon^\alpha h}(\mathbf{P}^\varepsilon - \mathbf{I}_\mathbf{N})$ and the matrix \mathbf{Q} of the limit Markov chain appearing in the Theorems 1.1 and 2.2.

Lemma 3.3. *Let $\mathbf{Q} = (q_{i,j})_{i,j=1}^n$ be the generator of the Markov chain Y on the state space \mathcal{M} defined in (1.10). Then there exist constants $\hat{C} = \hat{C}(\delta, h, R, \alpha)$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, every $1 \leq i, j \leq n$, $i \neq j$,*

$$\left| q_{i,j} - \frac{1}{\varepsilon^\alpha h} \sum_{y \in \mathcal{S}_j} p_{m_i, y}^\varepsilon \right| \leq \hat{C} \varepsilon^\alpha \quad (3.13)$$

and

$$\left| q_{i,i} - \frac{1}{\varepsilon^\alpha h} \sum_{y \in \mathcal{S}_j} (p_{m_i, y}^\varepsilon - \delta_{m_i}^y) \right| \leq \hat{C} \varepsilon^\alpha. \quad (3.14)$$

Proof. Let for definiteness $1 \leq i < j < n$. Then according to (2.8), (3.2) and (1.10)

$$\begin{aligned} \frac{1}{\varepsilon^\alpha h} \sum_{y \in \mathcal{S}_j} p_{m_i, y}^\varepsilon &= \frac{1}{\varepsilon^\alpha h} \sum_{y \in \mathcal{S}_j} \mathbb{P}\left(m_i - hU'(m_i) + \varepsilon^{\frac{1}{\alpha}} hL_1 \in I_y\right) \\ &= \frac{1}{\varepsilon^\alpha h} \mathbb{P}\left(m_i + \varepsilon^{\frac{1}{\alpha}} hL_1 \in [\mathfrak{s}_{j-1}, \mathfrak{s}_j]\right) \\ &= \frac{1}{2} \left(\frac{1}{(\mathfrak{s}_{j-1} - m_i)^\alpha} - \frac{1}{(\mathfrak{s}_j - m_i)^\alpha} \right) + \mathcal{O}(\varepsilon^\alpha) = q_{i,j} + \mathcal{O}(\varepsilon^\alpha). \end{aligned} \quad (3.15)$$

Analogously, for $j = n$, with the help of (2.10), (3.2) and (1.10) we get

$$\begin{aligned} \frac{1}{\varepsilon^\alpha h} \sum_{y \in \mathcal{S}_n} p_{m_i, y}^\varepsilon &= \frac{1}{\varepsilon^\alpha h} \sum_{y \in \mathcal{S}_n} \mathbb{P}\left(m_i - hU'(m_i) + \varepsilon^{\frac{1}{\alpha}} hL_1 \in I_y\right) \\ &= \frac{1}{\varepsilon^\alpha h} \mathbb{P}\left(m_i + \varepsilon^{\frac{1}{\alpha}} hL_1 \in [\mathfrak{s}_{n-1}, +\infty)\right) \\ &= \frac{1}{2} \frac{1}{(\mathfrak{s}_{j-1} - m_i)^\alpha} + \mathcal{O}(\varepsilon^\alpha) = q_{i,n} + \mathcal{O}(\varepsilon^\alpha), \end{aligned} \quad (3.16)$$

and for $i = j$

$$\begin{aligned}
\frac{1}{\varepsilon^\alpha h} \sum_{y \in \mathcal{S}_i} (p_{\mathbf{m}_i, y}^\varepsilon - \delta_{\mathbf{m}_i}^y) &= \frac{1}{\varepsilon^\alpha h} \left(-1 + \sum_{y \in \mathcal{S}_i} \mathbb{P}(\mathbf{m}_i - hU'(\mathbf{m}_i) + \varepsilon^{\frac{1}{\alpha}} h L_1 \in I_y) \right) \\
&= -\frac{1}{\varepsilon^\alpha h} \mathbb{P}(\mathbf{m}_i + \varepsilon^{\frac{1}{\alpha}} h L_1 \notin [\mathbf{s}_{i-1}, \mathbf{s}_i]) \\
&= -\frac{1}{2} \left(\frac{1}{(\mathbf{m}_i - \mathbf{s}_{i-1})^\alpha} + \frac{1}{(\mathbf{s}_j - \mathbf{m}_i)^\alpha} \right) + \mathcal{O}(\varepsilon^\alpha) = q_{i,i} + \mathcal{O}(\varepsilon^\alpha).
\end{aligned} \tag{3.17}$$

□

Now let us come to the investigation of the eigenvalues. We start with a well known result.

Proposition 3.4. *All eigenvalues of \mathbf{Q} and \mathbf{Q}^ε have non-positive real parts.*

Proof. This result follows from the Gershgorin circle theorem (see, for example, Theorem 1.11 in [46]). Indeed, since $q_{i,i} < 0$ for all $1 \leq i \leq n$, and $q_{x,x}^\varepsilon < 0$ for all $x \in \mathcal{S}$, as well as

$$\sum_{j \neq i} q_{i,j} = -q_{i,i} \quad \text{and} \quad \sum_{y \neq x} q_{x,y}^\varepsilon = -q_{x,x}^\varepsilon \tag{3.18}$$

all the Gershgorin circles are located in the negative complex half plane and touch the origin. In particular the principal eigenvalues equal to zero, $\lambda_1^\varepsilon = \lambda_1^{\mathbf{Q}} = 0$. □

The proof of Theorem 2.3 mainly consists in the investigation of the asymptotics of the characteristic polynomial $P^\varepsilon(\lambda) := \det(\mathbf{A}^\varepsilon(\lambda))$ in the limit $\varepsilon \rightarrow 0$. We will show that, after an appropriate scaling, it converges uniformly on a sufficiently large disc to the characteristic polynomial $P^{\mathbf{Q}}(\lambda) := \det(\mathbf{A}^{\mathbf{Q}}(\lambda))$ where $\mathbf{A}^{\mathbf{Q}}(\lambda) := \mathbf{Q} - \lambda \mathbf{I}_n$.

Proposition 3.5. *There exist constants $\varepsilon_0 > 0$, and $C = C(h, \delta, \alpha, R) > 1$ such that for every $0 < \varepsilon < \varepsilon_0$ and $\lambda \in \mathbb{C}$*

$$\left| (-\varepsilon^\alpha h)^{N-n} P^\varepsilon(\lambda) - P^{\mathbf{Q}}(\lambda) \right| \leq C(|\lambda| + C)^{n+1} \varepsilon^\alpha. \tag{3.19}$$

Before we start with the proof of this Proposition let us show how it implies the statements of Theorem 2.3.

Proof of Theorem 2.3. Both statements follow from Proposition 3.5 and the Hurwitz Theorem (see, for example, Theorem 1.3.8 in [40]):

Theorem 3.6 (Hurwitz). *Let $G \subset \mathbb{C}$ be a region and let $\{f_k\}_{k \geq 1}$ be a sequence of analytic functions on G that converges to a non-zero function f uniformly on every compact subset of G . Then $z_0 \in G$ is a zero of f with multiplicity m if and only if there exists a neighbourhood $H \subset G$ such that, in every ball $\{z \in \mathbb{C} : |z - z_0| \leq R\} \subset H$, each function f_k whose index exceeds some bound $k_0 = k_0(R)$ has exactly m zeros, counted according to their multiplicities.*

For the proof of (i) fix a closed ball centred around 0 that includes all the eigenvalues of \mathbf{Q} . Then Proposition 3.5 ensures the uniform convergence inside of this ball. Moreover, let a sequence $r_\varepsilon > 0$ be such that

$$\lim_{\varepsilon \rightarrow 0} r_\varepsilon = +\infty \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha r_\varepsilon^{n+1} = 0. \tag{3.20}$$

Then, Proposition 3.5 implies that

$$\lim_{\varepsilon \rightarrow 0} \sup_{|\lambda| \leq r_\varepsilon} |(-\varepsilon^\alpha h)^{N-n} P^\varepsilon(\lambda) - P^{\mathbf{Q}}(\lambda)| = 0 \tag{3.21}$$

and that proves the statement (ii). (Theorem 2.3) □

The rest of this section is devoted to the proof of Proposition 3.5. We start of with some preparations. Recall that the set \mathcal{S} consists of N elements, let

$$S_N = S_{|\mathcal{S}|} := \{\pi = (\pi_x)_{x \in \mathcal{S}} : \pi \text{ is bijective mapping } \mathcal{S} \rightarrow \mathcal{S}\} \tag{3.22}$$

be the set of all permutations of \mathcal{S} , and denote by \mathbf{id} the identity permutation. The Leibniz formula for the characteristic polynomial $P^\varepsilon(\lambda)$ gives

$$P^\varepsilon(\lambda) = \sum_{\pi \in S_N} \text{sgn}(\pi) \prod_{x \in \mathcal{S}} a_{x, \pi_x}^\varepsilon(\lambda). \tag{3.23}$$

Let us briefly explain the heuristics behind the following analysis of P^ε . Let the parameters h, δ, R and α be fixed. Clearly, in view of the asymptotics of the entries $a_{x,y}^\varepsilon$ as $\varepsilon \rightarrow 0$, for any fixed $\lambda \in \mathbb{C}$ we have $|P^\varepsilon(\lambda)| \rightarrow \infty$ as $\varepsilon \rightarrow 0$. With that in mind, let us omit for a moment the values $\tilde{a}_{x,y}^\varepsilon$ since they are of the order $\mathcal{O}(1)$, that is negligible in comparison to $\varepsilon^{-\alpha}$. Then one can think of $P^\varepsilon(\lambda)$ as of a polynomial of the variable $\varepsilon^{-\alpha}$ with coefficients depending on λ and terms of order $\mathcal{O}(1)$.

The degree of this polynomial does not exceed $N - n$ since only the rows indexed by the minima $\mathbf{m}_1, \dots, \mathbf{m}_n$ do not contain the term $(\varepsilon^\alpha h)^{-1}$. Hence we can write

$$P^\varepsilon(\lambda) \approx \sum_{j=0}^{N-n} c_j^\varepsilon(\lambda) (\varepsilon^\alpha h)^{-j}. \quad (3.24)$$

By a multiplication by the factor $(\varepsilon^\alpha h)^{N-n}$ we single out the coefficient $c_{N-n}^\varepsilon(\lambda)$ since all the other summands vanish in the limit $\varepsilon \rightarrow 0$. On the other hand, $c_{N-n}^\varepsilon(\lambda)$ contains expressions of the form $\sum_{y \in \mathcal{S}_j} \frac{1}{\varepsilon^\alpha h} p_{\mathbf{m}_i, y}^\varepsilon$, $1 \leq i, j \leq n$, which, as we saw in Lemma 3.3, are approximations of the entries $q_{i,j}$ of the generator matrix \mathbf{Q} . Finally, having that in mind we prove that $c_{N-n}^\varepsilon(\lambda)$ approximates $P^\mathbf{Q}(\lambda)$.

In view of the Leibniz formula (3.23) we should first turn to the permutations that contribute the terms of order $\mathcal{O}((\varepsilon^\alpha h)^{-(N-n)})$ to the polynomial P^ε . As we see from Lemma 3.2 (ii) and (iii), such permutations are precisely those which satisfy the condition $\pi_x = x$ or $\pi_x = y^*(x)$ for $x \in \mathcal{S} \setminus \mathcal{M}$. Consider the family of permutations

$$\Pi_0 := \{\pi = (\pi_x)_{x \in \mathcal{S}} \in S_N : \pi_x = x \text{ or } \pi_x = y^*(x) \text{ for all } x \in \mathcal{S} \setminus \mathcal{M}\}. \quad (3.25)$$

Lemma 3.7. *For every $i = 1, \dots, n$, every $y \in \mathcal{S}_i$ there exists a permutation $\pi \in \Pi_0$ such that*

$$\pi_{\mathbf{m}_i} = y. \quad (3.26)$$

Proof. Fix $y \in \mathcal{S}_i$. If $y = \mathbf{m}_i$, then we can choose $\pi = \text{id}$. Otherwise there exists a $k = k(y) \geq 1$ such that $((\mathbf{P}^0)^{(k)})_{y, \mathbf{m}_i} = 1$, that is the deterministic motion, if started in y , reaches the minimum \mathbf{m}_i in k steps. Recall that the mapping $T: \mathcal{S} \rightarrow \mathcal{S}$, $T(x) := y^*(x)$. Then the equality $((\mathbf{P}^0)^{(k)})_{y, \mathbf{m}_i} = 1$ is equivalent to

$$T^k y = \mathbf{m}_i. \quad (3.27)$$

Now we can define a permutation $\hat{\pi}: \mathcal{S} \rightarrow \mathcal{S}$ by

$$\hat{\pi}_y := Ty, \hat{\pi}_{Ty} = T^2 y, \dots, \hat{\pi}_{T^{k-1}y} = T^k y = \mathbf{m}_i, \hat{\pi}_{\mathbf{m}_i} = y \quad (3.28)$$

as well as

$$\hat{\pi}_z = z \quad \text{if } z \in \mathcal{S} \setminus \{y, Ty, \dots, T^k y\}, \quad (3.29)$$

It is clear that $\hat{\pi} \in \Pi_0$. □

Consider the family of vectors

$$\mathfrak{T}_n := \{(y_1, \dots, y_n) : y_i \in \mathcal{S}_i\}. \quad (3.30)$$

Lemma 3.8. *For every $(y_1, \dots, y_n) \in \mathfrak{T}_n$ there exists a permutation $\pi \in \Pi_0$ such that*

$$\pi_{\mathbf{m}_i} = y_i, \quad 1 \leq i \leq n. \quad (3.31)$$

Proof. Let $\pi_i, i = 1, \dots, n$, be the permutations constructed in the proof of Lemma 3.7. Note that each of them is non-identical only on the set \mathcal{S}_i . Hence a composition of $\pi_i, i = 1, \dots, n$, has the desired property. □

Lemma 3.9. *For every $(y_1, \dots, y_n) \in \mathfrak{T}_n$ there exists a permutation $\pi \in \Pi_0$ such that*

$$\pi_{\mathbf{m}_i} = y_j, \quad 1 \leq i, j \leq n. \quad (3.32)$$

Proof. Fix $(y_1, \dots, y_n) \in \mathfrak{T}_n$ and an associated permutation π from Lemma 3.8 such that $\pi_{\mathbf{m}_i} = y_i, i = 1, \dots, n$. Since no condition on the images of the states $\mathbf{m}_1, \dots, \mathbf{m}_n$ is imposed in the definition of Π_0 , we can permute them among each other to obtain $\pi \in \Pi_0$ with the required property. □

For the investigation of the determinant of the matrix $\mathbf{A}^\varepsilon(\lambda)$ we need to introduce the following disjoint decomposition of the set of all permutations S_N . For $p = 0, \dots, N - n$ let

$$\begin{aligned} \Pi_p := \{ \pi \in S_N : & \text{there are precisely } p \text{ states } x_1, \dots, x_p \in S \setminus \mathcal{M} \\ & \text{such that } \pi_{x_q} \notin \{x_q, y^*(x_q)\} \text{ for all } 1 \leq q \leq p \}. \end{aligned} \quad (3.33)$$

With this definition we can write

$$S_N = \bigsqcup_{p=0}^{N-n} \Pi_p. \quad (3.34)$$

Also note that the definition of Π_0 in (3.25) coincides with the one given in (3.33). For any $\pi \in S_N$ we also define

$$\begin{aligned} \mathcal{S}^1(\pi) &:= \{x \in S \setminus \mathcal{M} : \pi_x = x\}, \\ \mathcal{S}^2(\pi) &:= \{x \in S \setminus \mathcal{M} : \pi_x = y^*(x)\}, \\ \mathcal{S}^3(\pi) &:= S \setminus (\mathcal{S}^1(\pi) \sqcup \mathcal{S}^2(\pi)). \end{aligned} \quad (3.35)$$

Clearly these sets are disjoint and $\mathcal{S} = \bigsqcup_{i=1}^3 \mathcal{S}^i(\pi)$.

The following Lemma follows directly from the definition of Π_p .

Lemma 3.10. (i) For every $\pi \in \Pi_0$ we have $\mathcal{S}^3(\pi) = \mathcal{M}$.

(ii) For every $0 \leq p \leq N - n$ and every $\pi \in \Pi_p$ we have $|\mathcal{S}^1(\pi) \sqcup \mathcal{S}^2(\pi)| = N - n - p$. \square

Let us introduce functions $f_p : \pi \rightarrow \mathbb{N}_0$ that count the number of fixed points of $\pi \in \Pi_p$ on the set $S \setminus \mathcal{M}$, namely

$$f_p(\pi) := |\{x \in S \setminus \mathcal{M} : \pi_x = x\}| = |\mathcal{S}^1(\pi)|. \quad (3.36)$$

By definition of Π_p , it follows that f_p has values in $\{0, \dots, N - n - p\}$. We obtain a trivial equality

$$\prod_{x \in \mathcal{S}^1(\pi)} \left(-\frac{1}{\varepsilon^\alpha h} - \lambda + \tilde{a}_{x,x}^\varepsilon \right) = (-1)^{f_p(\pi)} \prod_{x \in \mathcal{S}^1(\pi)} \left(\frac{1}{\varepsilon^\alpha h} + (\lambda - \tilde{a}_{x,x}^\varepsilon) \right) \quad (3.37)$$

which allows us to write the characteristic polynomial P^ε as

$$\begin{aligned} P^\varepsilon(\lambda) &= \sum_{p=0}^{N-n} \left[\sum_{\pi \in \Pi_p} \text{sgn}(\pi) (-1)^{f_p(\pi)} \cdot \prod_{x \in \mathcal{S}^1(\pi)} \left(\frac{1}{\varepsilon^\alpha h} + (\lambda - \tilde{a}_{x,x}^\varepsilon) \right) \right. \\ &\quad \times \prod_{x \in \mathcal{S}^2(\pi)} \left(\frac{1}{\varepsilon^\alpha h} - \tilde{a}_{x,y^*(x)}^\varepsilon \right) \\ &\quad \left. \times \prod_{x \in \mathcal{S}^3(\pi)} a_{x,\pi_x}^\varepsilon \right]. \end{aligned} \quad (3.38)$$

Using Lemma 3.10 (ii) along with a simple straightforward calculation yields that there are numbers $\beta_{l,p,\pi}^\varepsilon(\lambda) = \mathcal{O}(1)$ such that

$$\prod_{x \in \mathcal{S}^1(\pi)} \left(\frac{1}{\varepsilon^\alpha h} + (\lambda - \tilde{a}_{x,x}^\varepsilon) \right) \prod_{x \in \mathcal{S}^2(\pi)} \left(\frac{1}{\varepsilon^\alpha h} - \tilde{a}_{x,x}^\varepsilon \right) = \sum_{l=0}^{N-n-p} \beta_{l,p,\pi}^\varepsilon(\lambda) \left(\frac{1}{\varepsilon^\alpha h} \right)^l. \quad (3.39)$$

As explained before P^ε can roughly be viewed as a polynomial in the variable $(\varepsilon^\alpha h)^{-1}$ and our aim is to single out the coefficient that belongs to the highest power.

After multiplying the polynomial P^ε by the factor $(\varepsilon^\alpha h)^{N-n}$ and taking Lemma 3.10 (i) into account, we split the sum in (3.38) up into three parts to get the representation

$$\begin{aligned} (\varepsilon^\alpha h)^{N-n} P^\varepsilon(\lambda) &= \sum_{\pi \in \Pi_0} \text{sgn}(\pi) (-1)^{f_0(\pi)} \prod_{i=1}^n a_{\mathbf{m}_i, \pi_{\mathbf{m}_i}}^\varepsilon(\lambda) \\ &\quad + \sum_{\pi \in \Pi_0} \text{sgn}(\pi) (-1)^{f_0(\pi)} \left(\sum_{l=0}^{N-n-1} \beta_{l,0,\pi}^\varepsilon(\lambda) (\varepsilon^\alpha h)^{N-n-l} \right) \prod_{i=1}^n a_{\mathbf{m}_i, \pi_{\mathbf{m}_i}}^\varepsilon(\lambda) \\ &\quad + \sum_{p=1}^{N-n} \sum_{\pi \in \Pi_p} \text{sgn}(\pi) (-1)^{f_p(\pi)} \left(\sum_{l=0}^{N-n-p} \beta_{l,p,\pi}^\varepsilon(\lambda) (\varepsilon^\alpha h)^{N-n-l} \right) \prod_{x \in \mathcal{S}^3(\pi)} a_{x,\pi_x}^\varepsilon. \end{aligned} \quad (3.40)$$

Let us continue the investigation of the permutations.

Lemma 3.11. Fix $\pi \in \Pi_0$ such that $\pi_{\mathbf{m}_i} \in \mathcal{S}_i$ for every $1 \leq i \leq n$. Then

$$\text{sgn}(\pi)(-1)^{f_0(\pi)} = (-1)^{N-n}. \quad (3.41)$$

Proof. Fix $1 \leq i \leq n$ and let $\pi_{\mathbf{m}_i} = y_i \in \mathcal{S}_i$. From this condition and from the definition of Π_0 it follows that for every $x \in \mathcal{S}_i$ we have $\pi_x \in \mathcal{S}_i$. Hence, we can write π as a product of n disjoint permutation cycles

$$\pi = \xi_1 \cdot \dots \cdot \xi_n, \quad (3.42)$$

where each cycle ξ_i acts only on \mathcal{S}_i . Define $k_i = k(y_i) \geq 0$ as the number of steps the deterministic motion needs to reach \mathbf{m}_i if started in y_i (see the proof of Lemma 3.7). Then we can write each cycle ξ_i as

$$\xi_i = \begin{cases} y_i \mapsto Ty_i \mapsto \dots \mapsto T^{k_i-1}y_i \mapsto \mathbf{m}_i \mapsto y_i, & k_i \geq 1, \\ \mathbf{m}_i \mapsto \mathbf{m}_i, & k_i = 0. \end{cases} \quad (3.43)$$

If case $k_i = 0$ then π acts as identity in \mathcal{S}_i , i.e. $\pi_x = x$ for all $x \in \mathcal{S}_i$. For the consistency of notation in this case, however, we do not omit the identity cycle ξ_i having in mind that the sign of an identity cycle is 1. Let $|\xi_i|$ denote the length of the i -th cycle. Then $|\xi_i| = k_i + 1$ and

$$\text{sgn}(\pi) = (-1)^{|\xi_1|-1} \cdot \dots \cdot (-1)^{|\xi_n|-1} = (-1)^{\sum_{i=1}^n k_i}. \quad (3.44)$$

On the other hand, from the definition of f_0 it follows that

$$f_0(\pi) = \sum_{i=1}^n (|\mathcal{S}_i| - (k_i + 1)) = N - n - \sum_{i=1}^n k_i \quad (3.45)$$

which proves the claim. \square

We construct a further decomposition of the set Π_0 . Let S_n denote the set of all permutations $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ and set

$$\Pi(\sigma) := \{\pi \in \Pi_0 : \pi_{\mathbf{m}_i} \in \mathcal{S}_{\sigma_i}, 1 \leq i \leq n\}. \quad (3.46)$$

Then it is easy to see that

$$\Pi_0 = \bigsqcup_{\sigma \in S_n} \Pi(\sigma). \quad (3.47)$$

Lemma 3.12. Let $\sigma \in S_n$. Then, for every $\pi \in \Pi(\sigma)$

$$\text{sgn}(\sigma)(-1)^{N-n} = \text{sgn}(\pi)(-1)^{f_0(\pi)}. \quad (3.48)$$

Proof. Let $\sigma = \zeta_1 \cdot \dots \cdot \zeta_L$ be a representation of σ as a product of $1 \leq L \leq n$ disjoint cycles. In particular, $|\zeta_1| + \dots + |\zeta_L| = n$.

From the definition of $\Pi(\sigma)$ it follows that there exists a collection of states $(y_1, \dots, y_n) \in \mathfrak{T}_n$ such that

$$\pi_{\mathbf{m}_1} = y_{\sigma_1} \in \mathcal{S}_{\sigma_1}, \dots, \pi_{\mathbf{m}_n} = y_{\sigma_n} \in \mathcal{S}_{\sigma_n}. \quad (3.49)$$

Now, every $\pi \in \Pi_0$ can be written as a composition $\pi = \pi_\sigma \circ \tilde{\pi}$ where $\tilde{\pi} \in \Pi_0$ such that $\tilde{\pi}_{\mathbf{m}_i} \in \mathcal{S}_i$ and where π_σ only acts on the states $y_1 = \tilde{\pi}_{\mathbf{m}_1}, \dots, y_n = \tilde{\pi}_{\mathbf{m}_n}$ according to the permutation σ . So let $\sigma(y_1, \dots, y_n)$ denote a cycle representation of π_σ . With the notations introduced in the previous lemmas we then can write π as

$$\pi = \sigma(y_1, \dots, y_n) \left(\mathbf{m}_1 \mapsto y_1 \mapsto \dots \mapsto T^{k_1-1}y_1 \right) \cdot \dots \cdot \left(\mathbf{m}_n \mapsto y_n \mapsto \dots \mapsto T^{k_n-1}y_n \right). \quad (3.50)$$

Keep in mind that the numbers k_i also depend on y_i and that one always has $T^{k_i}y_i = \mathbf{m}_i$. Since f_0 does not take $\mathbf{m}_1, \dots, \mathbf{m}_n$ into account we have $f_0(\pi) = f_0(\tilde{\pi})$. Combining that with the equality $\text{sgn}(\sigma) = \text{sgn}(\pi_\sigma)$ and Lemma 3.12 yields

$$\text{sgn}(\pi)(-1)^{f_0(\pi)} = \text{sgn}(\pi_\sigma) \text{sgn}(\tilde{\pi})(-1)^{f_0(\tilde{\pi})} = \text{sgn}(\sigma)(-1)^{N-n}. \quad (3.51)$$

\square

Lemma 3.11 allows us to rewrite the first summand in (3.40) as

$$\begin{aligned}
\sum_{\pi \in \Pi_0} \text{sgn}(\pi) (-1)^{f_0(\pi)} \prod_{i=1}^n a_{\mathfrak{m}_i, \pi_{\mathfrak{m}_i}}^\varepsilon(\lambda) &= \sum_{\sigma \in S_n} \sum_{\pi \in \Pi(\sigma)} \text{sgn}(\pi) (-1)^{f_0(\pi)} \prod_{i=1}^n a_{\mathfrak{m}_i, \pi_{\mathfrak{m}_i}}^\varepsilon(\lambda) \\
&= \sum_{\sigma \in S_n} \text{sgn}(\sigma) (-1)^{N-n} \sum_{\pi \in \Pi(\sigma)} \prod_{i=1}^n a_{\mathfrak{m}_i, \pi_{\mathfrak{m}_i}}^\varepsilon(\lambda) \\
&= (-1)^{N-n} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{y_1 \in \mathcal{S}_{\sigma_1}} \cdots \sum_{y_n \in \mathcal{S}_{\sigma_n}} \prod_{i=1}^n a_{\mathfrak{m}_i, y_i}^\varepsilon(\lambda) \\
&= (-1)^{N-n} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \sum_{y \in \mathcal{S}_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon(\lambda).
\end{aligned} \tag{3.52}$$

This representation makes clear why we also included the term $(-1)^{N-n}$ into the normalization factor for P^ε in Proposition 3.5.

The next lemma provides us with the crucial estimate.

Lemma 3.13. *There exist constants $\varepsilon_0 > 0$ and $\tilde{C} = \tilde{C}(h, \delta, R, \alpha, n) > 1$ such that for every $\lambda \in \mathbb{C}$ and every $0 < \varepsilon < \varepsilon_0$*

$$\left| \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \sum_{y \in \mathcal{S}_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon(\lambda) - \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma_i}^{\mathbf{Q}}(\lambda) \right| \leq \tilde{C}(\tilde{C} + |\lambda|)^{n-1} \varepsilon^\alpha. \tag{3.53}$$

Proof. First we need a generalization of the binomial formula that can be easily derived by a straightforward calculation. Fix numbers $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{R}$. Then

$$\prod_{i=1}^n (c_i + d_i) = \sum_{i=0}^n G_i \tag{3.54}$$

where

$$G_0 = \prod_{l=1}^n d_l \quad \text{and} \quad G_i = \sum_{1 \leq j_1 < \dots < j_i \leq n} \prod_{k \in \{j_1, \dots, j_i\}} c_k \prod_{\substack{l \in \{1, \dots, n\} \\ l \notin \{j_1, \dots, j_i\}}} d_l. \tag{3.55}$$

Now, abbreviate the left hand side of (3.53) with (LHS). We apply (3.54) with $c_i = \sum_{y \in \mathcal{S}_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon(\lambda) - a_{i, \sigma_i}^{\mathbf{Q}}(\lambda)$ and $d_i = a_{i, \sigma_i}^{\mathbf{Q}}(\lambda)$ to obtain

$$\begin{aligned}
(\text{LHS}) &\leq \sum_{\sigma \in S_n} \left| \prod_{i=1}^n \left(\sum_{y \in \mathcal{S}_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon(\lambda) - a_{i, \sigma_i}^{\mathbf{Q}}(\lambda) + a_{i, \sigma_i}^{\mathbf{Q}}(\lambda) \right) - \prod_{i=1}^n a_{i, \sigma_i}^{\mathbf{Q}}(\lambda) \right| \\
&= \sum_{\sigma \in S_n} \left| \prod_{i=1}^n c_i + \sum_{i=1}^{n-1} \sum_{1 \leq j_1 < \dots < j_i \leq n} \prod_{k \in \{j_1, \dots, j_i\}} c_k \prod_{\substack{l \in \{1, \dots, n\} \\ l \notin \{j_1, \dots, j_i\}}} d_l \right| \\
&\leq \sum_{\sigma \in S_n} \prod_{i=1}^n \left| \sum_{y \in \mathcal{S}_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon(\lambda) - a_{i, \sigma_i}^{\mathbf{Q}}(\lambda) \right| + \\
&\quad \underbrace{\sum_{\sigma \in S_n} \sum_{i=1}^{n-1} \sum_{1 \leq j_1 < \dots < j_i \leq n} \prod_{k \in \{j_1, \dots, j_i\}} \left| \sum_{y \in \mathcal{S}_{\sigma_k}} a_{\mathfrak{m}_k, y}^\varepsilon(\lambda) - a_{k, \sigma_k}^{\mathbf{Q}}(\lambda) \right| \prod_{\substack{l \in \{1, \dots, n\} \\ l \notin \{j_1, \dots, j_i\}}} |a_{l, \sigma_l}^{\mathbf{Q}}(\lambda)|}_{(*)}.
\end{aligned} \tag{3.56}$$

Since each row sum of $\mathbf{P}^\varepsilon - \mathbf{I}_N$ is equal to 0, we have for every $x \in \mathcal{S}$

$$\sum_{y \in \mathcal{S}_i} (\mathbf{P}^\varepsilon - \mathbf{I}_N)_{x, y} = - \sum_{\substack{j=1 \\ j \neq i}}^n \sum_{y \in \mathcal{S}_j} (\mathbf{P}^\varepsilon - \mathbf{I}_N)_{x, y}. \tag{3.57}$$

For $\sigma \in S_n$ define $F(\sigma) := \{1 \leq i \leq n : \sigma_i = i\}$ as the set of all the fixed points of σ . Then, with the help of Lemma 3.3 we derive for some constant $C > 0$

$$\begin{aligned}
& \sum_{\sigma \in S_n} \prod_{i=1}^n \left| \sum_{y \in S_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon(\lambda) - a_{i, \sigma_i}^{\mathbf{Q}}(\lambda) \right| \\
&= \sum_{\sigma \in S_n} \left[\prod_{\substack{i=1 \\ i \notin F(\sigma)}}^n \left| \sum_{y \in S_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon - a_{i, \sigma_i}^{\mathbf{Q}} \right| \cdot \prod_{\substack{i=1 \\ i \in F(\sigma)}}^n \left| - \sum_{\substack{j=1 \\ j \neq i}}^n \sum_{y \in S_j} a_{\mathfrak{m}_i, y}^\varepsilon - \lambda - \left(- \sum_{\substack{j=1 \\ j \neq i}}^n q_{i, j} - \lambda \right) \right| \right] \\
&= \sum_{\sigma \in S_n} \left[\prod_{\substack{i=1 \\ i \notin F(\sigma)}}^n \left| \sum_{y \in S_{\sigma_i}} \frac{1}{\varepsilon^\alpha h} p_{\mathfrak{m}_i, y}^\varepsilon - q_{i, \sigma_i} \right| \cdot \prod_{\substack{i=1 \\ i \in F(\sigma)}}^n \left| - \sum_{\substack{j=1 \\ j \neq i}}^n \sum_{y \in S_j} \frac{1}{\varepsilon^\alpha h} p_{\mathfrak{m}_i, y}^\varepsilon + \sum_{\substack{j=1 \\ j \neq i}}^n q_{i, j} \right| \right] \\
&\leq \sum_{\sigma \in S_n} (C\varepsilon^\alpha)^{n-|F(\sigma)|} (nC\varepsilon^\alpha)^{|F(\sigma)|} \\
&\leq n! (nC\varepsilon^\alpha)^n.
\end{aligned} \tag{3.58}$$

Similarly one can estimate the expression $(*)$ in (3.56) as follows:

$$\begin{aligned}
(*) &\leq \prod_{\substack{k \in \{j_1, \dots, j_i\} \\ k \notin F(\sigma)}} \left| \sum_{y \in S_k} \frac{1}{\varepsilon^\alpha h} p_{\mathfrak{m}_k, y}^\varepsilon - q_{k, \sigma_k} \right| \cdot \prod_{\substack{l \in \{1, \dots, n\} \\ l \notin \{j_1, \dots, j_i\} \\ l \notin F(\sigma)}} |q_{l, \sigma_l}| \\
&\quad \cdot \prod_{\substack{k \in \{j_1, \dots, j_i\} \\ k \in F(\sigma)}} \left| - \sum_{s \neq k} \sum_{y \in S_s} \frac{1}{\varepsilon^\alpha h} p_{\mathfrak{m}_k, y}^\varepsilon + \sum_{s \neq k} q_{k, s} \right| \cdot \prod_{\substack{l \in \{1, \dots, n\} \\ l \notin \{j_1, \dots, j_i\} \\ l \notin F(\sigma)}} |q_{l, l} - \lambda| \\
&\leq (C\varepsilon^\alpha)^{|\{k \in \{j_1, \dots, j_i\} : \sigma_k \neq k\}|} (nC\varepsilon^\alpha)^{|\{k \in \{j_1, \dots, j_i\} : \sigma_k = k\}|} \\
&\quad \times (\|\mathbf{Q}\|_\infty)^{|\{k \in \{1, \dots, n\} \setminus \{j_1, \dots, j_i\} : \sigma_k \neq k\}|} (\|\mathbf{Q}\|_\infty + |\lambda|)^{|\{k \in \{1, \dots, n\} \setminus \{j_1, \dots, j_i\} : \sigma_k = k\}|} \\
&\leq (nC\varepsilon^\alpha)^i (\|\mathbf{Q}\|_\infty + |\lambda|)^{n-i}.
\end{aligned} \tag{3.59}$$

We finish the proof by continuing in (3.56):

$$(\text{LHS}) \leq n! \cdot (nC\varepsilon^\alpha)^n + n! \sum_{i=1}^{n-1} \binom{n}{i} (nC\varepsilon^\alpha)^i (\|\mathbf{Q}\|_\infty + |\lambda|)^{n-i} \leq \tilde{C} \max_{1 \leq k \leq n-1} (\|\mathbf{Q}\|_\infty + |\lambda|)^k \varepsilon^\alpha. \tag{3.60}$$

Assigning \tilde{C} to be the maximum of \tilde{C} and $1 + \|\mathbf{Q}\|_\infty$ yields the result. \square

We now complete the proof of Proposition 3.5.

Proof of Proposition 3.5. First, we combine the equations (3.40) and (3.52) to obtain

$$\begin{aligned}
(-\varepsilon^\alpha h)^{N-n} P^\varepsilon(\lambda) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \sum_{y \in S_{\sigma_i}} a_{\mathfrak{m}_i, y}^\varepsilon \\
&\quad + (-1)^{N-n} \sum_{\pi \in \Pi_0} \text{sgn}(\pi) (-1)^{f_0(\pi)} \left(\sum_{l=0}^{N-n-1} \beta_{l,0,\pi}^\varepsilon(\lambda) (\varepsilon^\alpha h)^{N-n-l} \right) \prod_{i=1}^n a_{\mathfrak{m}_i, \pi_{\mathfrak{m}_i}}^\varepsilon(\lambda) \\
&\quad + (-1)^{N-n} \sum_{p=1}^{N-n} \sum_{\pi \in \Pi_p} \text{sgn}(\pi) (-1)^{f_p(\pi)} \left(\sum_{l=0}^{N-n-p} \beta_{l,p,\pi}^\varepsilon(\lambda) (\varepsilon^\alpha h)^{N-n-l} \right) \prod_{x \in S^3(\pi)} a_{x, \pi_x}^\varepsilon.
\end{aligned} \tag{3.61}$$

An application of Lemma 3.13 then yields

$$\begin{aligned}
& \left| (-\varepsilon^\alpha h)^{N-n} P^\varepsilon(\lambda) - P^{\mathbf{Q}}(\lambda) \right| \leq \tilde{C} (\tilde{C} + |\lambda|)^{n-1} \varepsilon^\alpha \\
&\quad + \sum_{\pi \in \Pi_0} \left| \sum_{l=0}^{N-n-1} \beta_{l,0,\pi}^\varepsilon(\lambda) (\varepsilon^\alpha h)^{N-n-l} \prod_{i=1}^n a_{\mathfrak{m}_i, \pi_{\mathfrak{m}_i}}^\varepsilon(\lambda) \right| \\
&\quad + \sum_{p=1}^{N-n} \sum_{\pi \in \Pi_p} \left| \sum_{l=0}^{N-n-p} \beta_{l,p,\pi}^\varepsilon(\lambda) (\varepsilon^\alpha h)^{N-n-l} \prod_{x \in S^3(\pi)} a_{x, \pi_x}^\varepsilon \right|.
\end{aligned} \tag{3.62}$$

We have to estimate the second and third summand. For this let us use Lemma 3.10 to renumber the states of $\mathcal{S}^1(\boldsymbol{\pi})$ and $\mathcal{S}^2(\boldsymbol{\pi})$ such that $\mathcal{S}^1(\boldsymbol{\pi}) = \{x_1, \dots, x_{f_p(\boldsymbol{\pi})}\}$ and $\mathcal{S}^2(\boldsymbol{\pi}) = \{x_{f_p(\boldsymbol{\pi})+1}, \dots, x_{N-n-p}\}$. Then, from (3.39) it is easy to see that the numbers $\beta_{l,p,\boldsymbol{\pi}}^\varepsilon(\lambda)$ are polynomials in λ of degree $|\mathcal{S}^1(\boldsymbol{\pi})| = f_p(\boldsymbol{\pi})$. Expanding them with respect to λ yields for $l = 0$

$$\beta_{0,p,\boldsymbol{\pi}}^\varepsilon(\lambda) = \prod_{\nu=1}^{f_p(\boldsymbol{\pi})} (\lambda - \tilde{a}_{x_\nu, x_\nu}^\varepsilon) \cdot \prod_{\nu=f_p(\boldsymbol{\pi})+1}^{N-n-p} (-\tilde{a}_{x_\nu, x_\nu}^\varepsilon) \quad (3.63)$$

as well as for $1 \leq l \leq N - n - p$

$$\beta_{l,p,\boldsymbol{\pi}}^\varepsilon(\lambda) = \sum_{1 \leq \mu_1 < \dots < \mu_l \leq N-n-p} \left(\prod_{\substack{\nu=1 \\ \nu \notin \{\mu_1, \dots, \mu_l\}}}^{f_p(\boldsymbol{\pi})} (\lambda - \tilde{a}_{x_\nu, x_\nu}^\varepsilon) \cdot \prod_{\substack{\nu=f_p(\boldsymbol{\pi})+1 \\ \nu \notin \{\mu_1, \dots, \mu_l\}}}^{N-n-p} (-\tilde{a}_{x_\nu, x_\nu}^\varepsilon) \right). \quad (3.64)$$

Then, by Lemma 3.2 and Lemma 3.1 we can find a constant $\bar{c} = \bar{c}(\alpha, h, \delta, R)$ such that for $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned} |\beta_{0,p,\boldsymbol{\pi}}^\varepsilon(\lambda)| &= \prod_{\nu=1}^{f_p(\boldsymbol{\pi})} |\lambda - \tilde{a}_{x_\nu, x_\nu}^\varepsilon| \cdot \prod_{\nu=f_p(\boldsymbol{\pi})+1}^{N-n-p} |\tilde{a}_{x_\nu, x_\nu}^\varepsilon| = \prod_{\nu=1}^{f_p(\boldsymbol{\pi})} \left| \lambda + \sum_{\substack{\kappa=1 \\ \kappa \neq \nu}}^N \tilde{a}_{x_\kappa, x_\kappa}^\varepsilon \right| \cdot \prod_{\nu=f_p(\boldsymbol{\pi})+1}^{N-n-p} |\tilde{a}_{x_\nu, x_\nu}^\varepsilon| \\ &\leq \left(\prod_{\nu=1}^{f_p(\boldsymbol{\pi})} (|\lambda| + N\bar{c}) \right) (\bar{c})^{N-n-p-f_p(\boldsymbol{\pi})} \\ &\leq (|\lambda| + N\bar{c})^{f_p(\boldsymbol{\pi})} (\bar{c})^{N-n-p-f_p(\boldsymbol{\pi})} \\ &\leq (|\lambda| + N\bar{c})^{N-n-p} \end{aligned} \quad (3.65)$$

and similarly

$$|\beta_{l,p,\boldsymbol{\pi}}^\varepsilon(\lambda)| \leq N!(|\lambda| + N\bar{c})^{N-n-p-l}, \quad 1 \leq l \leq N - n - p. \quad (3.66)$$

Also, note that for $\boldsymbol{\pi} \in \Pi_p$ we have $|\mathcal{S}^3(\boldsymbol{\pi})| = n + p$. Then

$$\left| \prod_{x \in \mathcal{S}^3(\boldsymbol{\pi})} a_{x, \pi_x}^\varepsilon(\lambda) \right| = \prod_{\substack{x \in \mathcal{S}^3(\boldsymbol{\pi}) \\ \pi_x = x}} |\lambda + \tilde{a}_{x, x}^\varepsilon| \prod_{\substack{x \in \mathcal{S}^3(\boldsymbol{\pi}) \\ \pi_x \neq x}} |a_{x, \pi_x}^\varepsilon| \leq (|\lambda| + N\bar{c})^{n+p}. \quad (3.67)$$

We finish the proof by continuing in (3.62) for sufficiently small ε and C big enough

$$\begin{aligned} &\left| (-\varepsilon^\alpha h)^{N-n} P^\varepsilon(\lambda) - P^Q(\lambda) \right| \\ &\leq \tilde{C}(\tilde{C} + |\lambda|)^{n-1} \varepsilon^\alpha \\ &\quad + (|\lambda| + N\bar{c})^n \sum_{\boldsymbol{\pi} \in \Pi_0} \sum_{l=0}^{N-n-1} N!(|\lambda| + N\bar{c})^{N-n-l} (\varepsilon^\alpha h)^{N-n-l} \\ &\quad + \sum_{p=1}^{N-n} (|\lambda| + N\bar{c})^{n+p} \sum_{\boldsymbol{\pi} \in \Pi_p} \sum_{l=0}^{N-n-p} N!(|\lambda| + N\bar{c})^{N-n-p-l} (\varepsilon^\alpha h)^{N-n-l} \\ &\leq \tilde{C}(\tilde{C} + |\lambda|)^{n-1} \varepsilon^\alpha \\ &\quad + (|\lambda| + N\bar{c})^n (N!)^2 \left[\sum_{l=0}^{\infty} (|\lambda| + N\bar{c})^l (\varepsilon^\alpha h)^l - 1 \right] \\ &\quad + (|\lambda| + N\bar{c})^n (N-n) (N!)^2 \left[\sum_{l=0}^{\infty} (|\lambda| + N\bar{c})^l (\varepsilon^\alpha h)^l - 1 \right] \\ &\leq \tilde{C}(\tilde{C} + |\lambda|)^{n-1} \varepsilon^\alpha + 2(|\lambda| + N\bar{c})^{n+1} (N!)^2 \varepsilon^\alpha h + 2(|\lambda| + N\bar{c})^{n+1} (N-n) (N!)^2 \varepsilon^\alpha h \\ &\leq C(C + |\lambda|)^{n+1} \varepsilon^\alpha. \end{aligned} \quad (3.68)$$

□

4 Eigenvectors. Proof of Theorem 2.4

For the investigation of the eigenvectors we will use some facts presented in [20] where the author considered the spectrum of the unscaled matrix $\mathbf{I}_N - \mathbf{P}^\varepsilon$. Let $\tilde{\lambda}_i = -\varepsilon^\alpha h \lambda_i^\varepsilon$, $1 \leq i \leq N$, be the eigenvalues of this matrix. Of course, the eigenspaces of \mathbf{Q}^ε and $\mathbf{I}_N - \mathbf{P}^\varepsilon$ coincide and Theorem 2.3 (i) and Proposition 3.4 imply that

$$\tilde{\lambda}_1^\varepsilon = 0, \quad \tilde{\lambda}_i^\varepsilon = \mathcal{O}(\varepsilon^\alpha), \quad 2 \leq i \leq n, \quad \text{and} \quad \operatorname{Re} \tilde{\lambda}_i^\varepsilon \geq 0, \quad 1 \leq i \leq n. \quad (4.1)$$

The proof of Theorem 2.4 needs some preparations. For a given subset $I \subset \mathcal{S}$ consider the first hitting time of I ,

$$\sigma_I^\varepsilon := \inf\{k \geq 0 : Z_k^\varepsilon \in I\}, \quad (4.2)$$

and, for convenience, the first return time to I ,

$$\tau_I^\varepsilon := \inf\{k \geq 1 : Z_k^\varepsilon \in I\}. \quad (4.3)$$

For $u \in \mathbb{C}$, $x \in \mathcal{S}$ and $I, J \subset \mathcal{S}$ consider the conditional Laplace transforms

$$G_{I,J}^{x,\varepsilon}(u) := \mathbb{E}_x \left[e^{u\tau_I^\varepsilon} \mathbf{1}_{\{\tau_I^\varepsilon \leq \tau_J^\varepsilon\}} \right] \quad (4.4)$$

and

$$K_{I,J}^{x,\varepsilon}(u) := \mathbb{E}_x \left[e^{u\sigma_I^\varepsilon} \mathbf{1}_{\{\sigma_I^\varepsilon \leq \sigma_J^\varepsilon\}} \right]. \quad (4.5)$$

These expressions are finite for every $u \in \mathbb{C}$ such that $\operatorname{Re} u \leq 0$ but they can be infinite if the real part of u is positive. We will discuss the finiteness of these Laplace transforms for certain special cases later.

Since for any initial state $x \in \mathcal{S} \setminus I$ we have the equality $\sigma_I^\varepsilon = \tau_I^\varepsilon$, it follows that

$$K_{I,J}^{x,\varepsilon}(u) = \begin{cases} G_{I,J}^{x,\varepsilon}(u), & x \notin I \cup J, \\ 1, & x \in I, \\ 0, & x \in J \setminus I. \end{cases} \quad (4.6)$$

In the following lemma we will prove another useful relation between these two functions (see Section 2 in [20]).

Lemma 4.1. *Fix an arbitrary $x \in \mathcal{S}$, subsets $I, J \subset \mathcal{S}$ and $u \in \mathbb{C}$ such that $G_{I,J}^{x,\varepsilon}(u)$ is finite. Then*

$$e^u \sum_{y \in \mathcal{S}} p_{x,y}^\varepsilon K_{I,J}^{y,\varepsilon}(u) = G_{I,J}^{x,\varepsilon}(u). \quad (4.7)$$

Proof. An application of the strong Markov property yields

$$\begin{aligned} G_{I,J}^{x,\varepsilon}(u) &= \mathbb{E}_x \left[e^{u\tau_I^\varepsilon} \mathbf{1}_{\{\tau_I^\varepsilon \leq \tau_J^\varepsilon\}} \right] = \sum_{y \in \mathcal{S}} \mathbb{E}_x \left[e^{u(1+\sigma_I^\varepsilon)} \mathbf{1}_{\{1+\sigma_I^\varepsilon \leq 1+\sigma_J^\varepsilon\}} | Z_1^\varepsilon = y \right] \mathbb{P}_x(Z_1^\varepsilon = y) \\ &= e^u \sum_{y \in \mathcal{S}} p_{x,y}^\varepsilon K_{I,J}^{y,\varepsilon}(u). \end{aligned} \quad (4.8)$$

□

Now let us introduce the following quantities. Define $T^{\max} \in \mathbb{N}$ to be the maximal number of steps the deterministic motion Z^0 needs to reach \mathcal{M} when starting somewhere in \mathcal{S} ,

$$T^{\max} = \max_{x \in \mathcal{S}} \min\{k \geq 1 : Z_k^0(x) \in \mathcal{M}\}. \quad (4.9)$$

Recall the definitions of the interval boundaries a_y and b_y of I_y and of the state $y^*(x)$ for a given $x \in \mathcal{S}$ and define

$$D := D(h, \delta) := \min_{x \in \mathcal{S}} \operatorname{dist} \left(x - hU'(x), [a_{y^*(x)}, b_{y^*(x)}] \right). \quad (4.10)$$

This quantity gives the minimal distance which the deterministic motion has to the interval boundaries after one time step and is crucial when it comes to comparing the trajectories of Z^0 and Z^ε . In particular, (3.5) implies that

$$\min_{x \in S} \mathbb{P}_x(Z_1^0 \neq Z_1^\varepsilon) = 1 - \mathbb{P}_x(Z_1^0 = Z_1^\varepsilon) = 1 - \mathbb{P}_x(Z_1^\varepsilon = y^*(x)) = p_{x, y^*(x)}^\varepsilon \geq \varepsilon^\alpha h D^{-\alpha}. \quad (4.11)$$

We denote $p^\varepsilon := \varepsilon^\alpha h D^{-\alpha}$, $p^\varepsilon = \mathcal{O}(\varepsilon^\alpha)$ as $\varepsilon \rightarrow 0$.

We are interested in the radius of convergence of the Laplace transforms $G_{I,J}^{x,\varepsilon}$ and $K_{I,J}^{x,\varepsilon}$ in dependence on ε and for the case $I, J \subset \mathcal{M}$. For this let

$$u_{\mathcal{M}}^\varepsilon := \sup \left\{ u \geq 0 : G_{\mathcal{M}, \mathcal{M}}^{x,\varepsilon}(u) = \mathbb{E}_x e^{u\tau_{\mathcal{M}}^\varepsilon} < \infty \text{ for all } x \in S \right\}. \quad (4.12)$$

Obviously, for $u \geq 0$ and $I \subset \mathcal{M}$ one always has $G_{I,\mathcal{M}}^{x,\varepsilon}(u) \leq G_{\mathcal{M},\mathcal{M}}^{x,\varepsilon}(u)$ so we can immediately conclude $G_{I,\mathcal{M}}^{x,\varepsilon}(u) < \infty$ for every $u \in \mathbb{C}$ with $0 \leq \operatorname{Re} u < u_{\mathcal{M}}^\varepsilon$.

Proposition 4.2. *We have*

$$\lim_{\varepsilon \rightarrow 0} u_{\mathcal{M}}^\varepsilon = \infty. \quad (4.13)$$

Proof. Consider $k \geq 1$ such that $T^{\max} + 1 \leq k \leq 2T^{\max}$. The occurrence of the event $\{\tau_{\mathcal{M}}^\varepsilon = k\}$ implies that the Markov chain Z^ε did not follow the deterministic motion Z^0 at at least one step on the time interval $[0, k]$, namely

$$\{\tau_{\mathcal{M}}^\varepsilon = k\} \subseteq \{Z_l^\varepsilon \neq Z_l^0 \text{ for at least one } 1 \leq l \leq k\}. \quad (4.14)$$

Similarly, for $2T^{\max} + 1 \leq k \leq 3T^{\max}$ the occurrence of $\{\tau_{\mathcal{M}}^\varepsilon = k\}$ implies that such a deviation happened at least twice and so forth. Hence, for $\mu \in \mathbb{N}$ and $\mu T^{\max} + 1 \leq k \leq (\mu + 1)T^{\max}$ one has

$$\{\tau_{\mathcal{M}}^\varepsilon = k\} \subseteq \{Z_l^\varepsilon \neq Z_l^0 \text{ for at least } \mu \text{ time instants } \{l_1, \dots, l_\mu\} \subset \{1, \dots, k\}\}. \quad (4.15)$$

Therefore for all $x \in S$, $\mu \in \mathbb{N}$ and $\mu T^{\max} + 1 \leq k \leq (\mu + 1)T^{\max}$ we find with the help of the Markov property that

$$\begin{aligned} \mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = k) &\leq \mathbb{P}_x\left(Z_l^\varepsilon \neq Z_l^0 \text{ for at least } \mu \text{ time instants } \{l_1, \dots, l_\mu\} \subset \{1, \dots, k\}\right) \\ &\leq \mathbb{P}_x\left(Z_l^\varepsilon \neq Z_l^0 \text{ for at least } \mu \text{ time instants } \{l_1, \dots, l_\mu\} \subset \{1, \dots, (\mu + 1)T^{\max}\}\right) \\ &\leq \sum_{l=\mu}^{(\mu+1)T^{\max}} \binom{(\mu+1)T^{\max}}{l} (p^\varepsilon)^l. \end{aligned} \quad (4.16)$$

We apply the well known estimate for the binomial coefficient $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$, $n \geq 1$, $1 \leq k \leq n$, and the bound $\mu \leq l \leq (\mu + 1)T^{\max}$ to get the estimate

$$\binom{(\mu+1)T^{\max}}{l} \leq C_1 K^\mu. \quad (4.17)$$

for some constants $C_1 > 0$ and $K > 1$. Then for some constant $C_2 > 0$:

$$\mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = k) \leq C_1 \left((\mu + 1)T^{\max} - \mu + 1 \right) K^\mu (p^\varepsilon)^\mu \leq C_2 \mu (K p^\varepsilon)^\mu. \quad (4.18)$$

For any $u \geq 0$ we set $C_3 = \sum_{k=1}^{T^{\max}} e^{uk} \mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = k)$ and obtain

$$\begin{aligned} \mathbb{E}_x e^{u\tau_{\mathcal{M}}^\varepsilon} &= \sum_{k=1}^{\infty} e^{uk} \mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = k) \\ &= C_3 + \sum_{\mu=1}^{\infty} \sum_{k=\mu T^{\max}+1}^{(\mu+1)T^{\max}} e^{uk} \mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = k) \\ &\leq C_3 + C_4 \sum_{\mu=1}^{\infty} e^{u(\mu+1)T^{\max}} \mu (K p^\varepsilon)^\mu. \end{aligned} \quad (4.19)$$

This series converges if $e^{uT^{\max}} K p^\varepsilon < 1$ or, equivalently, if $u < \frac{1}{T^{\max}} \ln\left(\frac{1}{p^\varepsilon}\right) - \ln(K)$. The claim of the proposition follows from the fact that p^ε is of order $\mathcal{O}(\varepsilon^\alpha)$. \square

From this proposition it follows that for a fixed $u_0 > 0$ one can always choose ε small enough such that $u_0 \leq u_{\mathcal{M}}^\varepsilon$ and therefore we can immediately deduce the following corollary that will provide us with an approximation of $G_{I,\mathcal{M}}^{x,\varepsilon}(u)$ with respect to u in a neighbourhood of 0.

Corollary 4.3. *Fix $u_0 > 0$ and $\varepsilon_0 > 0$ such that $u_0 \leq u_{\mathcal{M}}^\varepsilon$ and $0 < \varepsilon < \varepsilon_0$. Let $x \in \mathcal{S}$ and $I \subseteq \mathcal{M}$. Then there exists a constant $L > 0$, which can be chosen independently of ε and x , such that for every $u \leq u_0$*

$$|G_{I,\mathcal{M}}^{x,\varepsilon}(u) - G_{I,\mathcal{M}}^{x,\varepsilon}(0)| \leq L \cdot |u|. \quad (4.20)$$

Proof. The choice of ε guarantees that $G_{I,\mathcal{M}}^{x,\varepsilon}(u)$ is finite for $u \leq u_0$. We then use the Taylor expansion and obtain

$$|G_{I,\mathcal{M}}^{x,\varepsilon}(u) - G_{I,\mathcal{M}}^{x,\varepsilon}(0)| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left| \frac{d^n G_{I,\mathcal{M}}^{x,\varepsilon}(0)}{du^n} \right| \cdot |u|^n. \quad (4.21)$$

We have to estimate the expressions $\left. \frac{d^n G_{I,\mathcal{M}}^{x,\varepsilon}(u)}{du^n} \right|_{u=0}$ in the limit of small ε . An easy calculation yields

$$\begin{aligned} \left| \frac{d^n G_{I,\mathcal{M}}^{x,\varepsilon}(0)}{du^n} \right| &= \sum_{k=1}^{\infty} \left| \frac{d^n}{du^n} e^{uk} \right|_{u=0} \cdot \mathbb{P}_x(\tau_I^\varepsilon = k \leq \tau_{\mathcal{M}}^\varepsilon) \\ &= \sum_{k=1}^{\infty} k^n \mathbb{P}_x(\tau_I^\varepsilon = k \leq \tau_{\mathcal{M}}^\varepsilon) \\ &\leq \sum_{k=1}^{\infty} k^n \mathbb{P}_x(k \leq \tau_{\mathcal{M}}^\varepsilon) \\ &= \sum_{k=1}^{\infty} k^n \sum_{l=k}^{\infty} \mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = l). \end{aligned} \quad (4.22)$$

We proceed with similar arguments and the same constants C_2 and K as in the proof of Proposition 4.2. For a given $k \geq 1$ let $\mu_k \in \mathbb{N}$ be defined by the relation $\mu_k T^{\max} + 1 \leq k \leq (\mu_k + 1)T^{\max}$. Then,

$$\begin{aligned} \sum_{k=1}^{\infty} k^n \sum_{l=k}^{\infty} \mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = l) &\leq \sum_{k=1}^{\infty} k^n \sum_{\mu=\mu_k}^{\infty} \sum_{l=\mu T^{\max}+1}^{(\mu+1)T^{\max}} \mathbb{P}_x(\tau_{\mathcal{M}}^\varepsilon = l) \\ &\leq \sum_{k=1}^{\infty} k^n \sum_{\mu=\mu_k}^{\infty} C_2 \mu (Kp^\varepsilon)^\mu \\ &\leq C_3 \sum_{k=1}^{\infty} k^n \sum_{\mu=\mu_k}^{\infty} (2Kp^\varepsilon)^\mu \\ &= \frac{C_3}{1 - 2Kp^\varepsilon} \sum_{k=1}^{\infty} k^n (2Kp^\varepsilon)^{\mu_k}. \end{aligned} \quad (4.23)$$

We note that $\mu_k \leq \frac{k-1}{T^{\max}}$, abbreviate $r^\varepsilon := (2Kp^\varepsilon)^{\frac{1}{T^{\max}}}$, choose ε small enough such that $r^\varepsilon e^{u_0} < 1/2$ and find yet another constant $C_4 > 0$ such that

$$\begin{aligned} |G_{I,\mathcal{M}}^{x,\varepsilon}(u) - G_{I,\mathcal{M}}^{x,\varepsilon}(0)| &\leq C_4 \sum_{n=1}^{\infty} \frac{|u|^n}{n!} \sum_{k=0}^{\infty} k^n (r^\varepsilon)^k = C_4 \sum_{k=0}^{\infty} (r^\varepsilon)^k (e^{k|u|} - 1) \\ &= C_4 \frac{r^\varepsilon (e^{|u|} - 1)}{(1 - r^\varepsilon)(1 - r^\varepsilon e^{|u|})} \leq L|u| \end{aligned} \quad (4.24)$$

with a certain constant $L > 0$. □

Lemma 4.4. *Let \mathbf{A} be a $n \times n$ matrix such that $0 \in \sigma(\mathbf{A})$ and eigenspace corresponding to the eigenvalue 0 be one-dimensional. Let ψ be an eigenvector normalized such that $\|\psi\|_\infty := \max_{1 \leq j \leq n} |\psi_j| = 1$. Furthermore, let $\rho(\varepsilon)$, $\varepsilon > 0$, be a sequence of vectors such that $\lim_{\varepsilon \rightarrow 0} \|\rho(\varepsilon)\|_\infty = 0$ and assume that for every $\varepsilon > 0$ the equation $\mathbf{A}\mathbf{x} = \rho(\varepsilon)$ possesses a solution $\mathbf{x} = \psi(\varepsilon)$ such that $\|\psi(\varepsilon)\|_\infty = 1$. Then $\lim_{\varepsilon \rightarrow 0} \|\psi(\varepsilon) - \psi\|_\infty = 0$.*

Proof. The matrix \mathbf{A} defines a linear mapping from \mathbb{R}^n to \mathbb{R}^n . Let $E_1 := \ker(\mathbf{A})$ and $E_2 := \text{im}(\mathbf{A})$ such that $\mathbb{R}^n = E_1 \oplus E_2$. Then it is easy to see that the restriction $\mathbf{A}_2 := \mathbf{A}|_{E_2}$ of \mathbf{A} on E_2 a bijection from E_2 onto E_2 . Indeed, since every $x \in \mathbb{R}^n$ such that $\mathbf{A}x = 0$ is an element of E_1 we can deduce that $\ker(\mathbf{A}_2) = 0$ and hence \mathbf{A} is injective. We also know that for every $x \in E_2$, we can find $y = y_1 + y_2$, $y_1 \in E_1$, $y_2 \in E_2$ such that $x = \mathbf{A}y = \mathbf{A}y_2 = \mathbf{A}_2y_2$ which implies the surjectivity.

Now let $\psi(\varepsilon)$ be the normalized solution to $\mathbf{A}\psi(\varepsilon) = \rho(\varepsilon)$. We then can write $\psi(\varepsilon) = t(\varepsilon) + u(\varepsilon)$ for certain $t(\varepsilon) \in E_1$ and $u(\varepsilon) \in E_2$. Hence $\mathbf{A}u(\varepsilon) = \rho(\varepsilon)$. Since \mathbf{A} is invertible on E_2 and $(\mathbf{A})^{-1}|_{E_2} = (\mathbf{A}_2)|_{E_2}$ we can write $u(\varepsilon) = (\mathbf{A}_2)^{-1}\rho(\varepsilon)$ and therefore $\|\psi(\varepsilon) - t(\varepsilon)\|_\infty = \|u(\varepsilon)\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since $t(\varepsilon) \in E_1 = \{a\psi : a \in \mathbb{R}\}$ we can conclude by the normalization condition on $\psi(\varepsilon)$ that $\|\psi(\varepsilon) - \psi\|_\infty \rightarrow 0$ what finishes the proof. \square

Now we are able to finish the proof of Theorem 2.4.

Proof of Theorem 2.4:

For $i \leq 2 \leq n$, let u_i^ε be defined by $\tilde{\lambda}_i^\varepsilon = 1 - e^{-u_i^\varepsilon}$. By (4.1) it follows that $\lim_{\varepsilon \rightarrow 0} u_i^\varepsilon = 0$. We use a convenient representation of the eigenvectors of $\mathbf{I}_N - \mathbf{P}^\varepsilon$ obtained in [20, Lemma 4.1]. Namely, for any $\varepsilon > 0$, applying this lemma with $I = \emptyset$, $J = \mathcal{M}$ we get that there is a non-zero vector $(\psi_{\mathfrak{m}_1}^{\varepsilon,i}, \dots, \psi_{\mathfrak{m}_n}^{\varepsilon,i})$ such that the eigenvector of $\mathbf{I}_N - \mathbf{P}^\varepsilon$ is expressed in terms of the Laplace transforms $K_{\mathfrak{m}_j, \mathcal{M}}^{x, \varepsilon}$ as

$$\psi_x^{\varepsilon,i} = \sum_{j=1}^n \psi_{\mathfrak{m}_j}^{\varepsilon,i} K_{\mathfrak{m}_j, \mathcal{M}}^{x, \varepsilon}(u_i^\varepsilon), \quad x \in \mathcal{S}. \quad (4.25)$$

For definiteness let us normalize $\max_{1 \leq j \leq n} |\psi_{\mathfrak{m}_j}^{\varepsilon,i}| = 1$. Expanding the functions $u \mapsto K(u)$ with the help of Corollary 4.3 we get

$$K_{\mathfrak{m}_j, \mathcal{M}}^{x, \varepsilon}(u_i^\varepsilon) = \begin{cases} 1, & x = \mathfrak{m}_j, \\ \mathbb{P}_x(\sigma_{\mathfrak{m}_j}^\varepsilon \leq \sigma_{\mathcal{M}}^\varepsilon) + b_{x, \mathfrak{m}_j}(\varepsilon), & x \in \Omega_j, x \neq \mathfrak{m}_j, \\ \mathbb{P}_x(\sigma_{\mathfrak{m}_k}^\varepsilon \leq \sigma_{\mathcal{M}}^\varepsilon) + b_{x, \mathfrak{m}_k}(\varepsilon), & x \in \Omega_k, k \neq j, \end{cases} \quad (4.26)$$

where $\max_{x,k} |b_{x, \mathfrak{m}_k}(\varepsilon)| = \mathcal{O}(\varepsilon^\alpha)$. For $x \in \Omega_j$ we also have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(\sigma_{\mathfrak{m}_j}^\varepsilon \leq \sigma_{\mathcal{M}}^\varepsilon) = 1 \quad \text{as well as} \quad \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(\sigma_{\mathfrak{m}_k}^\varepsilon \leq \sigma_{\mathcal{M}}^\varepsilon) = 0, \quad k \neq j. \quad (4.27)$$

Recall that $\psi^{\varepsilon,i}$ satisfies the eigenvalue equation $\mathbf{A}^\varepsilon(\lambda_i^\varepsilon)\psi^{\varepsilon,i} = 0$. From this system, let us single out n equations indexed by $\mathfrak{m}_1, \dots, \mathfrak{m}_n$:

$$\sum_{j=1}^n \psi_{\mathfrak{m}_j}^{\varepsilon,i} \sum_{y \in \mathcal{S}} (\mathbf{A}^\varepsilon(\lambda_i^\varepsilon))_{\mathfrak{m}_k, y} \cdot \left(\mathbb{P}_y(\sigma_{\mathfrak{m}_j}^\varepsilon \leq \sigma_{\mathcal{M}}^\varepsilon) + b_{y, \mathfrak{m}_j}(\varepsilon) \right) = 0, \quad k = 1, \dots, n. \quad (4.28)$$

Abbreviate $\mathbf{A}^Q(\lambda) := \mathbf{Q} - \lambda \mathbf{I}_n$ and recall the following limits:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sum_{y \in \mathcal{S}_j} (\mathbf{A}^\varepsilon(\lambda))_{\mathfrak{m}_k, y} &= (\mathbf{A}^Q(\lambda))_{k, j}, \quad j = 1, \dots, n, \\ \lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha h \lambda_i^\varepsilon &= \lambda_i^Q. \end{aligned} \quad (4.29)$$

Thus we can denote

$$\rho_k^i(\varepsilon) = \sum_{j=1}^n (\mathbf{A}^Q(\lambda_i^Q))_{k, j} \psi_{\mathfrak{m}_j}^{\varepsilon,i}, \quad k = 1, \dots, n. \quad (4.30)$$

where $\max_k |\rho_k^i(\varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$. Recall that the eigenspace of λ_i^Q is one dimensional and apply Lemma 4.4. That proves the assertion of the theorem. \square

References

- [1] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, second edition, 2009.
- [2] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani. The mechanism of stochastic resonance. *Journal of Physics A*, 14:453–457, 1981.

- [3] N. Berglund. Kramers' law: Validity, derivations and generalisations. *Markov Processes and Related Fields*, 19(3):459–490, 2013.
- [4] N. Berglund and B. Gentz. The Eyring–Kramers law for potentials with nonquadratic saddles. *Markov Processes and Related Fields*, 16(3):549–598, 2010.
- [5] H. Bergström. On some expansions of stable distribution functions. *Arkiv för Matematik*, 2(4):375–378, 1952.
- [6] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability and low lying spectra in reversible Markov chains. *Communications in Mathematical Physics*, 228:219–255, 2002.
- [7] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. *Journal of the European Mathematical Society*, 6(4):399–424, 2004.
- [8] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues. *Journal of the European Mathematical Society*, 7(1):69–99, 2005.
- [9] T. Burghoff. *Spectral Properties of a Discrete Lévy-driven Metastable System*. PhD thesis, Friedrich Schiller University Jena, 2014.
- [10] V. A. Buslov and K. A. Makarov. Hierarchy of time scales in the case of weak diffusion. *Theor. Math. Phys.*, 76(2):818–826, 1988.
- [11] V. A. Buslov and K. A. Makarov. Life times and lower eigenvalues of an operator of small diffusion. *Matematicheskie Zametki*, 51(1):20–31, 1992.
- [12] K. Byl and R. Tedrake. Metastable walking machines. *The International Journal of Robotics Research*, 28(8):1040–1064, 2009.
- [13] M. K. Cameron. Computing Freidlin's cycles for the overdamped Langevin dynamics. Application to the Lennard–Jones-38 cluster. *Journal of Statistical Physics*, 152(3):493–518, 2013.
- [14] S. Cerrai. *Second order PDE's in finite and infinite dimension. A probabilistic approach*, volume 1762 of *Lecture notes in mathematics*. Springer, 2001.
- [15] A. V. Chechkin, V. Yu. Gonchar, J. Klafter, R. Metzler, and L. V. Tanatarov. Lévy flights in a steep potential well. *Journal of Statistical Physics*, 115(5–6):1505–1535, 2004.
- [16] T.-S. Chiang, C.-R. Hwang, and S.-J. Sheu. Diffusion for global optimization in \mathbb{R}^n . *SIAM Journal on Control and Optimization*, 25(3):737–753, 1987.
- [17] P. D. Ditlevsen. Observation of α -stable noise induced millennial climate changes from an ice record. *Geophysical Research Letters*, 26(10):1441–1444, May 1999.
- [18] A. Dubkov and B. Spagnolo. Langevin approach to Lévy flights in fixed potentials: Exact results for stationary probability distributions. *Acta Physica Polonica B*, 38(5):1745–1758, 2007.
- [19] B. Dybiec, I. M. Sokolov, and A. V. Chechkin. Stationary states in single-well potentials under symmetric Lévy noises. *Journal of Statistical Mechanics: Theory and Experiment*, page P07008, 2010.
- [20] M. Eckhoff. *Capacity and low lying spectra of attractive Markov chains*. PhD thesis, University of Potsdam, 2000.
- [21] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften*. Springer, second edition, 1998.
- [22] A. Friedman. The asymptotic behavior of the first real eigenvalue of a second order elliptic operator with a small parameter in the highest derivatives. *Indiana University Mathematics Journal*, 22(10):1005–1015, 1973.
- [23] A. Galves, E. Olivieri, and M. E. Vares. Metastability for a class of dynamical systems subject to small random perturbations. *The Annals of Probability*, 15(4):1288–1305, 1987.
- [24] S. Herrmann, P. Imkeller, I. Pavlyukevich, and D. Peithmann. *Stochastic Resonance: A Mathematical Approach in the Small Noise Limit*, volume 194 of *AMS Mathematical Surveys and Monographs*. American Mathematical Society, 2014.
- [25] R. Holley, S. Kusuoka, and D. Stroock. Asymptotics of the spectral gap with applications to the theory of simulated annealing. *Journal of Functional Analysis*, 83(2):333–347, 1989.

- [26] C.-R. Hwang and S.-J. Sheu. Large-time behavior of perturbed diffusion Markov processes with applications to the second eigenvalue problem for Fokker-Planck operators and simulated annealing. *Acta Applicandae Mathematicae*, 19(3):253–295, 1990.
- [27] P. Imkeller and I. Pavlyukevich. Metastable behaviour of small noise Lévy-driven diffusions. *ESAIM: Probability and Statistics*, 12:412–437, 2008.
- [28] R. Z. Khas'minski. On positive solutions of the equation $\mathfrak{A}u + Vu = 0$. *Theory of Probability and its Applications*, 4(3):309–318, 1959.
- [29] C. Kipnis and C. M. Newman. The metastable behavior of infrequently observed, weakly random, one-dimensional diffusion processes. *SIAM Journal on Applied Mathematics*, 45(6):972–982, 1985.
- [30] V. N. Kolokoltsov. *Semiclassical analysis for diffusions and stochastic processes*, volume 1724 of *Lecture Notes in Mathematics*. Springer, 2000.
- [31] V. N. Kolokol'tsov and K. A. Makarov. Asymptotic spectral analysis of a small diffusion operator and the life times of the corresponding diffusion process. *Russian Journal of Mathematical Physics*, 4(3):341–360, 1996.
- [32] H. A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, 7:284–304, 1940.
- [33] A. M. Kulik. Exponential ergodicity of the solutions to SDE's with a jump noise. *Stochastic Processes and their Applications*, 119(2):602–632, 2009.
- [34] K. A. Makarov. Division of the spectrum of an elliptic operator associated with “small diffusion”. *Vestnik Leningrad University. Mathematics*, 18(1):27–36, 1985.
- [35] B. J. Matkowsky and Z. Schuss. Eigenvalues of the Fokker-Planck operator and the approach to equilibrium for diffusions in potential fields. *SIAM Journal on Applied Mathematics*, 40:242–254, 1981.
- [36] R. Metzler, A. V. Chechkin, and J. Klafter. Lévy statistics and anomalous transport: Lévy flights and subdiffusion. In M. A. Meyers, editor, *Computational Complexity. Theory, Techniques, and Applications*, pages 1724–1745. Springer, 2012.
- [37] P. Metzner, C. Schütte, and E. Vanden-Eijnden. Transition path theory for Markov jump processes. *Multiscale Modeling & Simulation*, 7(3):1192–1219, 2009.
- [38] C. Nicolis. Stochastic aspects of climatic transitions — responses to periodic forcing. *Tellus*, 34:1–9, 1982.
- [39] E. Olivieri and M. E. Vares. *Large deviations and metastability*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2003.
- [40] Q. I. Rahman and G. Schmeisser. *Analytic Theory of Polynomials*. London Mathematical Society Monographs. Clarendon Press, 2002.
- [41] G. Samorodnitsky and M. Grigoriu. Tails of solutions of certain nonlinear stochastic differential equations driven by heavy tailed Lévy motions. *Stochastic Processes and their Applications*, 105(1):69–97, 2003.
- [42] Z. Schuss. *Theory and applications of stochastic processes. An analytical approach*, volume 170 of *Applied Mathematical Sciences*. Springer, 2010.
- [43] Z. Schuss and B. J. Matkowsky. The exit problem: A new approach to diffusion across potential barriers. *SIAM Journal on Applied Mathematics*, 36(3):604–623, 1979.
- [44] C. Schütte and M. Sarich. *Metastability and Markov state models in molecular dynamics: modeling, analysis, algorithmic approaches*, volume 24 of *Courant Lecture Notes*. American Mathematical Society, 2013.
- [45] P. N. V. Tu. *Dynamical systems: An introduction with applications in Economics and Biology*. Springer-Verlag, second edition, 1994.
- [46] R. S. Varga. *Matrix Iterative Analysis*. Springer Series in Computational Mathematics. Springer, 2009.